An interesting phenomenon

\( L(1,i) = \text{probability of } i \text{ to be last vertex visited before all vertices are visited starting at vertex 1} \)

\[ L(1,i) = \frac{1}{n - 1} \]

High dimensions

\( d = 2: \)
\[ \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2} \]

\( d = 4: \)
\[ \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1 \]

\( d = \text{large number}: \)
\[ \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{d}{2}} \]

Volume of sphere

Cartesian: volume
\[ \int_{x_1}^{x_{n+1}} \int_{x_2}^{x_{n+1}} \int_{x_3}^{x_{n+1}} \ldots \int_{x_n}^{x_{n+1}} dx_1 \cdots dx_n \cdots dx_1 \]

(1)

Polar: volume
\[ \int_{r=0}^{1} r^{d-1} d\Omega \]

\[ \int_{r=0}^{1} r^{d-1} d\Omega = \frac{1}{d} \int_{r=0}^{1} r^{d-1} dr \]

(2)

Cartesian: \( I(d) = \int_{x_1}^{x_{n+1}} \int_{x_2}^{x_{n+1}} \ldots \int_{x_n}^{x_{n+1}} e^{-\frac{x_1^2 + x_2^2 + \ldots + x_n^2}{2}} \]

(3)

Polar: \( I(d) = \int_{r=0}^{1} r^{d-1} e^{-\frac{r^2}{2}} d\Omega \]

(4)
For (3), we continue to have:

\[ I(d) = \left[ \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx \right]^d \quad (5) \]

Since the integral can be considered as a constant multiplying integration of the pdf of Gaussian distribution with \( \sigma = 1 \) from \(-\infty\) to \(\infty\) whose value is 1, (5) is evaluated to \((\sqrt{2\pi})^d\).

For (4), from formula sheet, we have

\[ \int_0^{\infty} r^{d-1} e^{-\frac{r^2}{2}} \, dr = \frac{1}{2} \cdot \Gamma \left( \frac{d}{2} \right) \quad (6) \]

Divide (4) by (6), we get

\[ \int_s d\Omega = \frac{(\sqrt{2\pi})^d}{\frac{1}{2} \cdot \Gamma \left( \frac{d}{2} \right)} \quad (7) \]

Plug (7) into (2), we get

\[ V(d) = \frac{1}{d} \frac{(\sqrt{2\pi})^d}{\frac{1}{2} \cdot \Gamma \left( \frac{d}{2} \right)} \quad (8) \]

**Note:**
We suspect that the RHS of equation (6) might be incorrect. This may lead to equation (8)’s inconsistency with Professor Hopcroft’s original solution, where

\[ V(d) = \frac{1}{d} \frac{\pi^{\frac{d}{2}}}{\frac{1}{2} \cdot \Gamma \left( \frac{d}{2} \right)} \]

A clarification will be provided in the next lecture.

Q: For what value of d is V(d) max?
A: d = 11

**Generating random points in high dimensions**
In dimension $d$, distance between any two points is:

$$dist(x_1, x_2) = \sqrt{\sum_{i=1}^{d} (x_{1i} - x_{2i})^2}$$

To be continued...