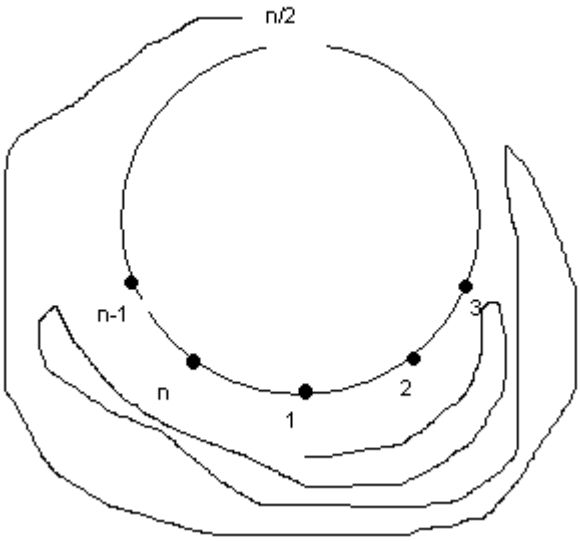
An interesting phenomenon

L(1,i) = probability of i to be last vertex visited before all vertices are visited starting at vertex 1



L(1,i) = $\dfrac{1}{n-1}$

High dimensions

d = 2: $\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \frac{\sqrt{2}}{2}$

d = 4: $\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2} = 1$

d = large number: $\sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2 + \ldots + (\frac{1}{2})^2} = \sqrt{d(\frac{1}{2})^2} = \frac{\sqrt{d}}{2}$

Volume of sphere

Cartesian: volume = $\int_{x_1=-1}^{x_1=1}\int_{x_2=-\sqrt{1-x_1^2}}^{x_2=\sqrt{1-x_1^2}} \ldots\ldots \int_{x_d=-\sqrt{1-x_1^2-x_2^2-\ldots}}^{x_d=\sqrt{1-x_1^2-x_2^2-\ldots}} dx_d \ldots dx_2 dx_1$ (1)

Polar: volume = $\int_\Omega \int_{r=0}^{1} r^{d-1} d\Omega\, dr = \int_s d\Omega \int_{r=0}^{1} r^{d-1} dr = \frac{1}{d}\int_s d\Omega$ (2)

Cartesian: $I(d) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} e^{-\frac{x_1^2+x_2^2+\ldots+x_d^2}{2}} dx_d \ldots dx_2 dx_1$ (3)

Polar: $I(d) = \int_\Omega \int_r r^{d-1} e^{-\frac{r^2}{2}} d\Omega\, dr = \int_\Omega d\Omega \int_r r^{d-1} e^{-\frac{r^2}{2}} dr$ (4)

For (3), we continue to have:

$$I(d) = \left[\int_{-\infty}^{\infty} e^{-\frac{x_1^2}{2}} dx \right]^d \qquad (5)$$

Since the integral can be considered as a constant multiplying integration of the pdf of Gaussian distribution with $\sigma = 1$ from $-\infty$ to ∞ whose value is 1, (5) is evaluated to $(\sqrt{2\pi})^d$.

For (4), from formula sheet, we have

$$\int_0^{\infty} r^{d-1} e^{-\frac{r^2}{2}} dr = \frac{1}{2} \cdot \Gamma(\frac{d}{2}) \qquad (6)$$

Divide (4) by (6), we get

$$\int_s d\Omega = \frac{(\sqrt{2\pi})^d}{\frac{1}{2}\Gamma(\frac{d}{2})} \qquad (7)$$

Plug (7) into (2), we get

$$V(d) = \frac{1}{d} \frac{(\sqrt{2\pi})^d}{\frac{1}{2}\Gamma(\frac{d}{2})} \qquad (8)$$

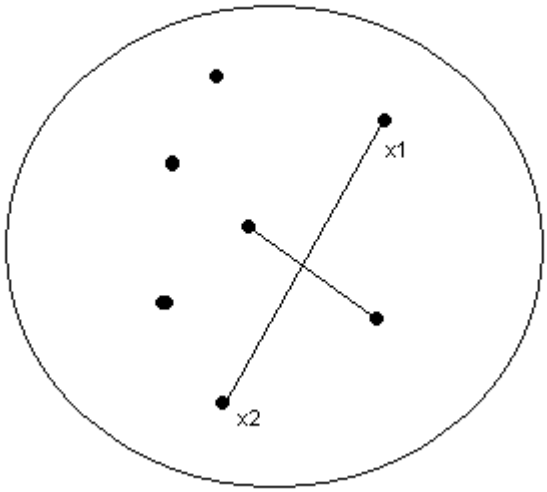
<u>Note:</u>
We suspect that the RHS of equation (6) might be incorrect. This may lead to equation (8)'s inconsistency with Professor Hopcroft's original solution, where

$$V(d) = \frac{1}{d} \frac{\pi^{\frac{d}{2}}}{\frac{1}{2}\Gamma(\frac{d}{2})}$$

A clarification will be provided in the next lecture.

Q: For what value of d is V(d) max?
A: $d \approx 11$

<u>Generating random points in high dimensions</u>



In dimension d, distance between any two points is:

$$dist(x_1, x_2) = \sqrt{\sum_{i=1}^{d} (x_{1i} - x_{2i})^2}$$

To be continued…