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Spectral Analysis of Random Graphs (4/6/07)

Random Matrix

$$P(\lambda) = \frac{2}{\pi}\sqrt{1-\lambda^2}$$

- To show that the two are equal, find the moments of both sides and see that they are the same.

Right Side

- Normalize by a constant $2\sigma\sqrt{n}$
- Let $c(k)$ be the k[th] moment of $P(\lambda) = \frac{2}{\pi}\sqrt{1-\lambda^2}$
- $c(k) = \frac{2}{\pi}\int_{-1}^{1} x^k \sqrt{1-x^2}\,dx$
- $c(k) = \begin{cases} 0 & k\,odd \\ \frac{1}{k+2}\frac{1}{2^{k-1}}\binom{k}{k/2} & k\,even \end{cases}$

Left Side

- Let $m(k)$ be the k[th] moment of $P(\lambda)$
- Normalize $\rightarrow \frac{\lambda_j}{2\sqrt{n}}$
- $m(k) = E\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\lambda_j}{2\sqrt{n}}\right)^k\right] = \frac{1}{2^k}\frac{1}{n^{1+k/2}}E\left[\sum_{j=1}^{n}\lambda^k\right]$
- By linear algebra: $m(k) = \frac{1}{2^{nk}}\frac{1}{n^{1+k/2}}E\left[trace(A^k)\right]$
- $A_{i,j}^k$ is the sum over all paths of length k from i to j of product of edge labels
- Since we want the trace, we're interested in $A_{i,i}^k$.
- $m(k) = \frac{1}{2^{nk}}\frac{1}{n^{1+k/2}}E\left[A_{1,1}^k + A_{2,2}^k + \ldots + A_{n,n}^k\right]$
- We're only interested in paths in which each edge that occurs occurs at least twice. Since if it only occurs once, the expected value of that edge is: $\frac{1}{2}*1+\frac{1}{2}*-1=0$
- What is the number of distinct vertices on a path of length k?
- If you have to go through each edge twice then the maximum number of vertices is k/2. This will only occur if the first time you go through an edge, you go to a new vertex. Therefore the only paths of length k that have k/2 vertices are depth first searches.
- number of paths = (number of types of trees)*(number of trees of each type)

> Recall that σ equals 1 when the adjacency matrix contains ±1 with probability ½.
> $\therefore \frac{\lambda_j}{2\sigma\sqrt{n}} = \frac{\lambda_j}{2\sqrt{n}}$

- Since depth first searches are isomorphic, they can be represented as a string of balanced parenthesis, which can be counted using the Catalan numbers.

$$number\ of\ \ paths = Cat_{k/2} * n^{k/2}$$

$$Cat_{k/2} = \frac{1}{\frac{k}{2}+1}\binom{k}{k/2}$$

$$m(k) = \frac{1}{2^k}\frac{1}{n^{1+k/2}}n\frac{1}{k/2+1}\binom{k}{k/2}n^{k/2}$$

$$m(k) = \frac{1}{k+2}\frac{1}{2^{k-1}}\binom{k}{k/2}$$

$$m(k) = \frac{1}{k+2}\frac{1}{2^{k-1}}\binom{k}{k/2} = c(k)$$

$$\therefore P(\lambda) = \frac{2}{\pi}\sqrt{1-\lambda^2}$$

Higher Dimensions :

1. Consider a square in different dimensions (i.e. square, cube, etc)
We know that volume of unit square (i.e. in 2D) = 1
Volume of unit cube (i.e. in 3D) = 1
In general, volume = 1 (for any number of dimensions)

2. Volume of unit circle = π
Volume of unit sphere = 4/3 * π
But what happens in higher dimensions?
We expect the volume to keep increasing to infinity (or at least some non zero multiple of π), but in reality the volume goes to zero as the number of dimensions increases. (Proof of this in the next lecture).