CS 485 Lecture 26 (March 28, 2007)

John Allen, Hyun Joo Noh

5 Axioms

1) Pagerank satisfies the 5 Axioms

2) Any ordering that satisfies axioms is equivalent to pagerank.

How can this be? If you change restart rate, will it keep the same ordering?

-> 5 Axioms only apply to strongly connected graphs with unweighted edges.

Ranking: voting, movies, restaurants, WWW pages, etc

Arrow: an economist who came up with axioms for voting systems.

eg)	A	40%
	B	35%
	C	25%

Is A the winner? What if everyone who voted for C would vote for B over A? Then A would have 40%, B would have 60%. Is A still the winner?

- Removing irrelevant factors shouldn't affect the result.

eg)	4 votes	A > B > C
	3 votes	B > C > A
	2 votes	C > A > B

Least people voted for C as their first choice. So drop C, and give the 2 votes to A instead. Is A the winner?

But 5 people (2nd and 3rd cases) prefer C over A, while only 4 people (1st case) prefer A over C. Maybe C should win?

- Ranking is very complex, and it is hard to make it fair.

<u>Pagerank satisfies Axiom #2</u>

(Axiom 2: Adding a self-loop to *v* does not change ordering of any other vertices but may improve ranking for v)

<u>Proof:</u>

Adjacency matrix A:

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & & & & \\ & \dots & & & \\ \dots & & & & \\ \dots & & & & \\ a_{n1} & & & & \end{pmatrix}$$

Normalize rows:

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ \dots & & & & \\ & \dots & & & \\ \dots & & & & \\ \dots & & & & \\ b_{n1} & & & & \end{pmatrix}$$

Transpose:

$$B^T = \begin{pmatrix} 0 & b_{21} & b_{31} & \dots & b_{n1} \\ b_{12} & b_{22} & b_{32} & \dots & b_{n2} \\ \dots & & & & \\ & \dots & & & \\ \dots & & & & \\ \dots & & & & \\ b_{1n} & & & & \end{pmatrix}$$

Looking for vector $r = \begin{pmatrix} r_1 \\ r_2 \\ .. \\ r_n \end{pmatrix}$ such that $B^T r = r$

Change 0 in matrix A to 1, then renormalize 1st column of B^T.
All columns other than the 1st column in B^T stay the same.

Assume m = number of 1's in 1st row of A.

$$B'^{T}r' = \begin{pmatrix} 1/(m+1) & b_{21} & b_{31} & \dots & b_{n1} \\ b_{22}*m/(m+1) & b_{32} & \dots & b_{n2} & \\ b_{13}*m/(m+1) & & & & \\ \dots & & & & \\ \dots & & & & \\ \dots & & & & \\ b_{1n}*\ m/(m+1) & & & & \end{pmatrix} \begin{pmatrix} r_1 \\ r_2*m/(m+1) \\ \dots \\ \dots \\ \dots \\ \dots \\ r_n*m/(m+1) \end{pmatrix}$$

$$= \begin{pmatrix} r_1/(m+1) + r1*m/(m+1) \\ r_2*m/(m+1) \\ r_3*m/(m+1) \\ \dots \\ \dots \\ \dots \\ r_n*m/(m+1) \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2*m/(m+1) \\ r_3*m/(m+1) \\ \dots \\ \dots \\ \dots \\ r_n*m/(m+1) \end{pmatrix} = r'$$

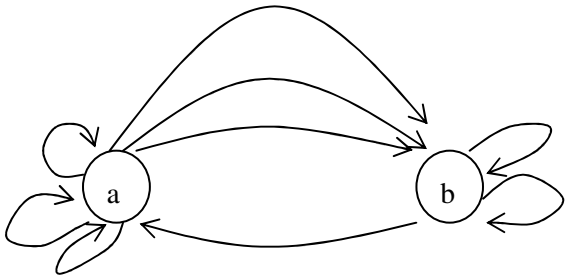
For r_2 to r_n, everything was multiplied by the same constant m/(m+1)
=> ordering hasn't been changed for vertices other than *v*.

<u>Any ordering that satisfies 5 axioms is equivalent to pagerank</u>

<u>Proof:</u>

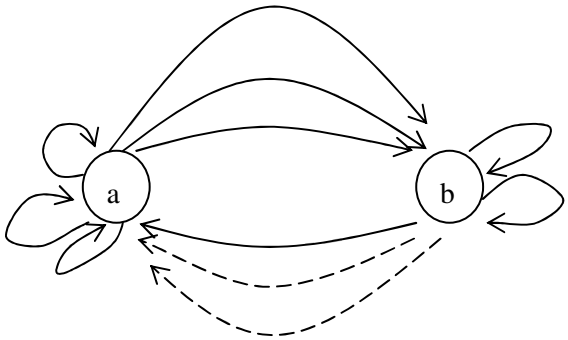
Proof outline: Select two vertices *a* & *b* and show that any ordering satisfying axioms preserves order of *a* and *b*

Remove each vertex *v* in some manner that does not change order of *a* and *b*.
The remaining graph would look something like this,



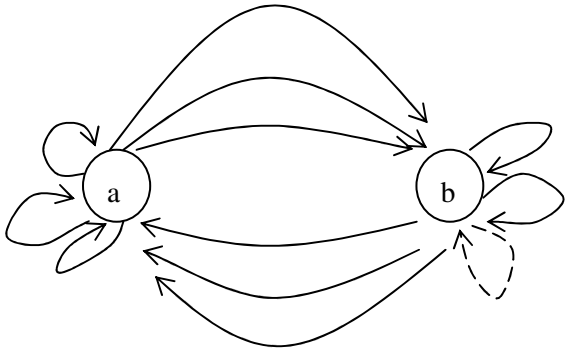
with some links from *a* to *b*, some links from *b* to *a*, and some self-loops.

Equalize links $a \to b$ and $b \to a$ without changing the order



Equalize number of self-loops.

(In our example, this may improve rank of b.)



By Axiom 1, ranking of *a* and *b* must be the same now.

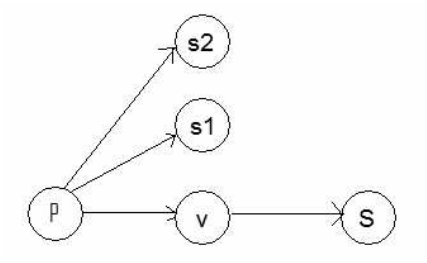
(In our example, *b* must have had equal or lower rank than *a* before equalizing self-loops.)

To convince you that the above process is possible we will prove that part of the process is performable without altering the ordering of a and b.

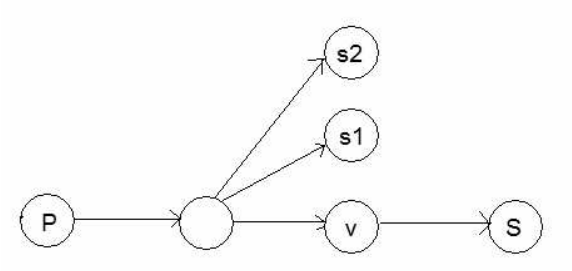
Lemma 4.2

Let v be a vertex with in and out degree 1, Successor s, and predecessor p. Then if the ranking satisfies the 5 axioms we can remove v and replace edges (p,v) and (v,s) with edge (p,s).

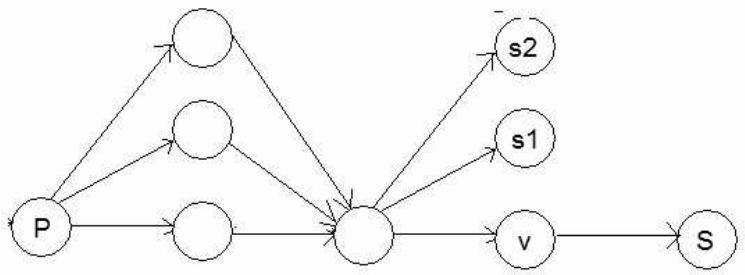
Starting case:



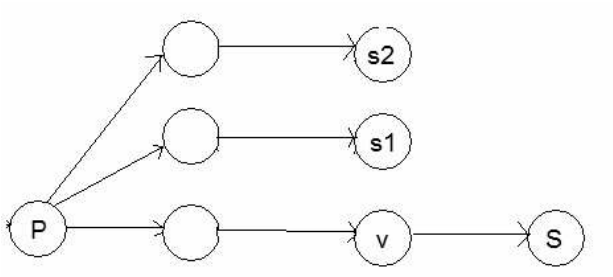
Axiom 3 – vote by committee allows us to insert a committee of 1 for P



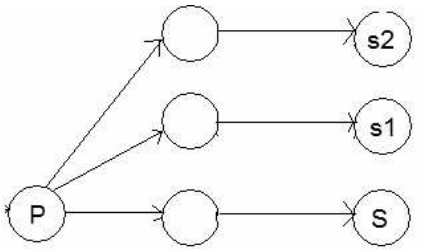
Axiom 3 – vote by committee allows us to insert a committee of 3 for P



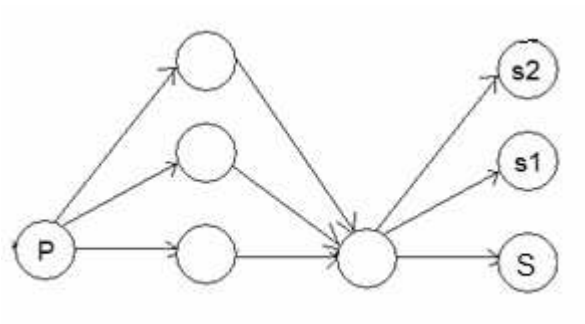
Since the members of the new committee of three are all ranked equally, Axiom 5 allows us to change the graph in this way



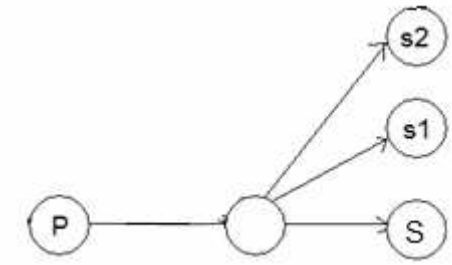
Axiom 3 implies that v can now be removed as it essentially a committee of one for its predecessor



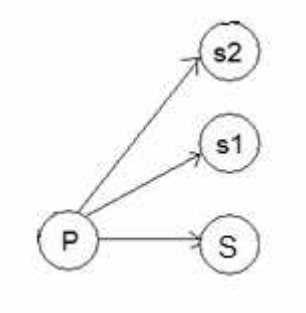
Axiom 5 implies that we can insert a node in the following way without changing the ranking.



Axiom 3 implies that the committee of 3 for P can be removed



Axiom 3 implies that the committee of 1 for P can be removed



Therefore we are able to remove any vertex with in and out degree 1 without changing the relative rankings of any pair of nodes (a,b) in the graph.