5 Axioms
1) Pagerank satisfies the 5 Axioms
2) Any ordering that satisfies axioms is equivalent to pagerank.

How can this be? If you change restart rate, will it keep the same ordering?
-> 5 Axioms only apply to strongly connected graphs with unweighted edges.

Ranking: voting, movies, restaurants, WWW pages, etc

Arrow: an economist who came up with axioms for voting systems.

eg) A 40%
    B 35%
    C 25%
Is A the winner? What if everyone who voted for C would vote for B over A? Then A would have 40%, B
would have 60%. Is A still the winner?
    ✗ Removing irrelevant factors shouldn’t affect the result.

eg) 4 votes A > B > C
    3 votes B > C > A
    2 votes C > A > B
Least people voted for C as their first choice. So drop C, and give the 2 votes to A instead. Is A the
winner?
But 5 people (2nd and 3rd cases) prefer C over A, while only 4 people (1st case) prefer A over C. Maybe C
should win?
    ✗ Ranking is very complex, and it is hard to make it fair.

Pagerank satisfies Axiom #2
(Axiom 2: Adding a self-loop to v does not change ordering of any other vertices but may improve
ranking for v)
Proof:
Adjacency matrix A:
\[
A = \begin{pmatrix}
0 & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
a_{n1} & & & & \\
\end{pmatrix}
\]

Normalize rows:

\[
B = \begin{pmatrix}
0 & b_{12} & b_{13} & \cdots & b_{1n} \\
b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
b_{n1} & & & & \\
\end{pmatrix}
\]

Transpose:

\[
B^T = \begin{pmatrix}
0 & b_{21} & b_{31} & \cdots & b_{n1} \\
b_{12} & b_{22} & b_{32} & \cdots & b_{n2} \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
b_{1n} & & & & \\
\end{pmatrix}
\]

Looking for vector \( r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix} \) such that \( B^T r = r \)

Change 0 in matrix A to 1, then renormalize 1st column of \( B^T \).
All columns other than the 1st column in \( B^T \) stay the same.

Assume \( m = \) number of 1's in 1st row of A.
\[ B^T r' = \begin{pmatrix} \frac{1}{m+1} & b_{21} & b_{31} & \ldots & b_{n1} \\ b_{22} \frac{m}{m+1} & b_{32} & \ldots & b_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} \frac{m}{m+1} & b_{2n} & \ldots & b_{nn} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \frac{m}{m+1} \\ \vdots \\ r_n \frac{m}{m+1} \end{pmatrix} \]

\[
= \begin{pmatrix} r_1 \frac{1}{m+1} + r_1 \frac{m}{m+1} \\ r_2 \frac{m}{m+1} \\ r_3 \frac{m}{m+1} \\ \vdots \\ r_n \frac{m}{m+1} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \frac{m}{m+1} \\ r_3 \frac{m}{m+1} \\ \vdots \\ r_n \frac{m}{m+1} \end{pmatrix} = r'
\]

For \( r_2 \) to \( r_n \), everything was multiplied by the same constant \( \frac{m}{m+1} \)

\( \Rightarrow \) ordering hasn’t been changed for vertices other than \( v \).

Any ordering that satisfies 5 axioms is equivalent to pagerank

Proof:

Proof outline: Select two vertices \( a \) & \( b \) and show that any ordering satisfying axioms preserves order of \( a \) and \( b \)

Remove each vertex \( v \) in some manner that does not change order of \( a \) and \( b \).

The remaining graph would look something like this,

with some links from \( a \) to \( b \), some links from \( b \) to \( a \), and some self-loops.
Equalize links \( a \to b \) and \( b \to a \) without changing the order.

![Diagram showing equalization of links](image)

Equalize number of self-loops.
(In our example, this may improve rank of \( b \).)

![Diagram showing equalization of self-loops](image)

By Axiom 1, ranking of \( a \) and \( b \) must be the same now.
(In our example, \( b \) must have had equal or lower rank than \( a \) before equalizing self-loops.)
To convince you that the above process is possible we will prove that part of the process is performable without altering the ordering of $a$ and $b$.

**Lemma 4.2**

Let $v$ be a vertex with in and out degree 1, Successor $s$, and predecessor $p$. Then if the ranking satisfies the 5 axioms we can remove $v$ and replace edges $(p,v)$ and $(v,s)$ with edge $(p,s)$.

Starting case:

Axiom 3 – vote by committee allows us to insert a committee of 1 for $P$

Axiom 3 – vote by committee allows us to insert a committee of 3 for $P$

Since the members of the new committee of three are all ranked equally, Axiom 5 allows us to change the graph in this way
Axiom 3 implies that v can now be removed as it essentially a committee of one for its predecessor.

Axiom 5 implies that we can insert a node in the following way without changing the ranking.

Axiom 3 implies that the committee of 3 for P can be removed.
Axiom 3 implies that the committee of 1 for P can be removed.

Therefore we are able to remove any vertex with in and out degree 1 without changing the relative rankings of any pair of nodes (a,b) in the graph.