Random Walks on Undirected Graphs (3/16/07)

Theorem: Commute time \((u,v) = 2mk_{av}^f\).

Proof: Insert current \(d_i\) into each vertex \(i\) where \(d_i\) is the degree of node \(i\) and withdraw \(2m\) amps current from vertex \(j\).

\[
d_i = \sum_{k \text{adjoi}} (v_{ij} - v_{kj}) = d_i v_{ij} - \sum_k v_{kj}
\]

\[
v_{ij} = 1 + \frac{1}{d_i} \sum_k v_{kj}
\]

\[
= \sum_k \frac{1}{d_i} (1 + v_{kj})
\]

\[
h_{ij} = \sum_k \frac{1}{d_i} (1 + h_{kj})
\]

\[
v_{ij} = h_{ij}
\]

Instead of removing current from \(j\) remove it from \(i\).

\[
v_{ji} = h_{ji}
\]

Reverse current in above step, we get:

\[
-v_{ji} = h_{ji}
\]

\[
h_{ji} = v_{ij}
\]

Two Scenarios

1. Put current in proportional to degree at every node and take out at \(j\).
2. Put current in at \(i\) and take it out at all individual ones.

Overall we can say we applied \(2m\) current at \(i\) and took it out at \(j\).
Superimpose both situations:

\[ v_{ij} = k_{ij}^e + h_{ji} = 2mk_{ij}^e \]

commute time (i, j) = \( 2mk_{ij}^e \)

\( k_{ij}^e \) number of edges

**Directed Walks (PageRank)**

Some problems:
- If you start at a, you never reach 1, 2, 3 (not strongly connected)
- Where do you go once you are at b? (no out edges)

Solutions:
- Put self-loop at b? Not a very good solution.
- With probability \( p \), have each node take a random jump to a random vertex.
  E.g. \( p=0.15 \); Prob(c goes to a) = 0.85, Prob(c goes to a vertex taken at random) = 0.15
- Essentially, create a new vertex labeled “Restart” and give it an edge from every other vertex, such that the edge to “Restart” is taken with probability \( p \).
- PageRank of “Restart” vertex is guaranteed to be \( p \).
- This is not typically done in practice, however.

Start at 1. What is the stationary probability of each vertex? The cycles don’t converge!

Adding edge \( a \) solves the problem.