Probabilistic Interpretation of Current

Previously proven relation:
\[ v_x = \sum_{y \text{adj} x} v_y P_{xy} \]

Definition:
\[ u_x = \text{number of visits to } x \text{ on trip from } a \text{ to } b \]
\[ u_b = 0 \]

Postulate to be derived:
\[ \frac{u_x}{C_x} = \sum_{y \text{adj} x} \frac{u_y}{C_y} P_{xy} \]

Derivation:
\[ u_x = \sum_{y \text{adj} x} u_y P_{yx} \text{ where } P_{yx} = \frac{C_{yx}}{C_y} = \frac{C_{xy}}{C_y} = \frac{C_x}{C_y} P_{xy} \]
\[ u_x = \sum_{y \text{adj} x} \frac{u_y C_x}{C_y} P_{xy} \]
\[ \frac{u_x}{C_x} = \sum_{y} \frac{u_y}{C_y} P_{xy} \]

we can sum over all \( y \) and just set probabilities for nonadjacent nodes to zero

Now we can arbitrarily set \( v_b = 0 \) as a reference point. We are still free to set the voltage of \( v_a \) and will set it to \( \frac{u_a}{C_a} \). This implies \( \forall x \mid v_x = \frac{u_x}{C_x} \) since both are harmonic functions and have the same boundary conditions. What does this force the value of the current \( I \) to be?
\[ I = \sum y i_{ay} \]
\[ = \sum_y (v_a - v_y) C_{ay} \]
\[ = v_a \sum y C_{ay} - \sum y v_y C_{ay} \]
\[ = v_a C_a - \sum y v_y C_y \frac{C_{ya}}{C_y} \]
\[ = u_a - \sum y u_y P_{ya} = 1 + \text{number of returns to } a - \text{number of returns to } a = 1 \]

Now we can interpret the meaning of current in the context of random walks:
\[
i_{xy} = (v_x - v_y)C_{xy} = \left(\frac{u_x}{C_x} - \frac{u_y}{C_y}\right)C_{xy} = u_x \frac{C_{xy}}{C_x} - u_y \frac{C_{xy}}{C_y} = u_x P_{yx} - u_y P_{xy}
\]

Which can be read as the number of times the walk reaches \(x\) and goes to \(y\), minus the number of times the walk reaches \(y\) and goes to \(x\), which is precisely the net number of crossings of the edge \((x,y)\) from \(x\) to \(y\).

**Effective Resistance and Escape Probability**

**Def:** Escape probability is the probability that a random walk started at \(a\) reaches \(b\) without first returning to \(a\).

Consider a grid of points as shown on the left where \(b\) is the boundary of the graph, shown in red. We want to consider the escape probability in the limit as the distance from \(b\) to \(a\) approaches infinity.

We will show that \(P_{\text{escape}} = \frac{C_{\text{eff}}}{C_a}\), by calculating the current at \(a\) which will give us an expression for the escape probability and the effective conductance, which we can equate to reach the above relation.

Set \(v_a = 1\), \(v_b = 0\).

\[
i_a = \sum_{y \text{ adj to } a} (v_a - v_y) c_{ay}
\]

\[
= \sum_{y \text{ adj to } a} \left(1 - v_y\right) \frac{c_{ay}}{c_a}
\]

\[
= c_a \left[\sum_{y \text{ adj to } a} \frac{c_{ay}}{c_a} - \sum_{y \text{ adj to } a} v_y \cdot P_{ay}\right]
\]

\[
= c_a \left[1 - \sum_{y \text{ adj to } a} v_y \cdot P_{ay}\right]
\]

\(v_y \cdot P_{ay} = \text{Probability of reaching } a \text{ before } b \text{ for walk starting at } y\)
\[ \sum_{y \text{ adj to } a} v_y \cdot P_{ay} = \text{Probability of returning to } a \text{ on walk from } a \text{ to } b \text{ (i.e. not escaping)} \]

\[ 1 - \sum_{y \text{ adj to } a} v_y \cdot P_{ay} = \text{Probability of escape} = P_{escape} \]

\[ \therefore i_a = c_a \cdot P_{escape} \]

\[ i_a = v_a \cdot C_{eff} = C_{eff} \]

Equating the two expressions we have: \( c_a \cdot P_{escape} = C_{eff} \) or \( P_{escape} = \frac{C_{eff}}{c_a} \)

1 Dimension:

Replace all edges by 1\( \Omega \) resistors.

\[ R_{eff} = \frac{k}{2} \text{ (we have 2 paths for } b = k \text{) and } C_{eff} = \frac{2}{k} \]

Therefore, \( \lim_{k \to \infty} C_{eff} = 0 \). This means that if we start at the origin we will always come back, in the one dimensional case.

2 Dimensions:

To do this we will short all resistors in concentric squares in the lattice. We will thus determine a lower bound for \( R_{eff} \) and show that it is still infinite.

For the first few squares we have \( R_{eff} = 1/4, 1/12, 1/20 \ldots \)

\[ R_{eff} \geq \frac{1}{4} + \frac{1}{12} + \frac{1}{20} + \ldots \]

\[ R_{eff} \geq \frac{1}{4} \left[ 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots \right] \geq \frac{1}{4} \ln n \]

This implies that \( \lim_{n \to \infty} C_{eff} = 0 \) and that \( \lim_{n \to \infty} P_{escape} = 0 \)

3 Dimensions:
In the 3-dimensional case we can construct a tree rooted at the origin that branches. If we do this properly we get the sum as an upper bound:

\[
R_{\text{eff}} \leq \frac{1}{3} \left[ 1 + \frac{2}{3} + \left( \frac{2}{3} \right)^2 + \cdots \right] = \frac{1}{3} \left[ \frac{1}{1 - \frac{2}{3}} \right] = 1
\]

⇒ Probability of escape is some number between 0 and 1.