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Probabilistic Interpretation of Current

Previously proven relation :

$$v_x = \sum_{y\, adj\, x} v_y P_{xy}$$

Definition :

$u_x =$ number of visits to x on trip from a to b

$u_b = 0$

Postulate to be derived :

$$\frac{u_x}{C_x} = \sum_y \frac{u_y}{C_y} P_{xy}$$

Derivation :

$$u_x = \sum_{y\, adj\, x} u_y P_{yx} \quad \text{where } P_{yx} = \frac{C_{yx}}{C_y} = \frac{C_{xy}}{C_y} = \frac{C_x}{C_y} P_{xy}$$

$$u_x = \sum_{y\, adj\, x} u_y \frac{C_x}{C_y} P_{xy}$$

$$\frac{u_x}{C_x} = \sum_y \frac{u_y}{C_y} P_{xy}$$ we can sum over all y and just set probabilities for nonadjacent nodes to zero

Now we can arbitrarily set $v_b = 0$ as a reference point. We are still free to set the voltage of v_a and will set it to u_a/C_a. This implies $\forall x \mid v_x = u_x/C_x$ since both are harmonic functions and have the same boundary conditions. What does this force the value of the current I to be?

$$I = \sum_y i_{ay}$$

$$= \sum_y (v_a - v_y) C_{ay}$$

$$= v_a \sum_y C_{ay} - \sum_y v_y C_{ay}$$

$$= v_a C_a - \sum_y v_y C_y \frac{C_{ya}}{C_y}$$

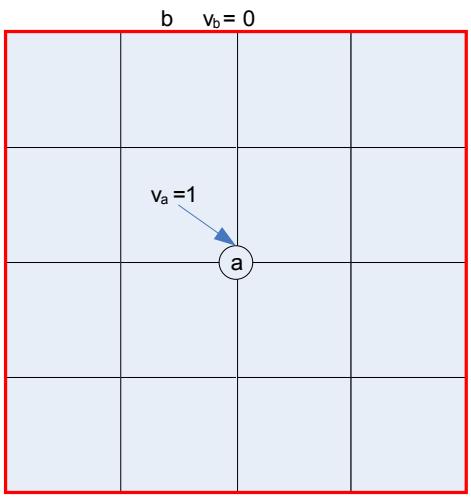
$$= u_a - \sum_y u_y P_{ya} = 1 + \text{number of returns to a} - \text{number of returns to a} = 1$$

Now we can interpret the meaning of current in the context of random walks:

$$i_{xy} = (v_x - v_y)C_{xy} = \left(\frac{u_x}{C_x} - \frac{u_y}{C_y}\right)C_{xy} = u_x\frac{C_{xy}}{C_x} - u_y\frac{C_{yx}}{C_y} = u_xP_{xy} - u_yP_{yx}$$

Which can be read as the number of times the walk reaches x and goes to y, minus the number of times the walk reaches y and goes to x, which is precisely the net number of crossings of the edge (x,y) from x to y.

Effective Resistance and Escape Probability



Def: Escape probability is the probability that a random walk started at a reaches b without first returning to a.

Consider a grid of points as shown on the left where b is the boundary of the graph, shown in red. We want to consider the escape probability in the limit as the distance from b to a approaches infinity.

We will show that $P_{escape} = \frac{C_{eff}}{C_a}$, by calculating the current at a which will give us an expression for the escape probability and the effective conductance, which we can equate to reach the above relation.

Set $v_a = 1, v_b = 0$.

$$i_a = \sum_{\substack{y \text{ adj} \\ \text{to } a}} (v_a - v_y)c_{ay}$$

$$= \sum_{\substack{y \text{ adj} \\ \text{to } a}} (1 - v_y)\frac{c_{ay}}{c_a}c_a$$

$$= c_a\left[\sum_{\substack{y \text{ adj} \\ \text{to } a}} \frac{c_{ay}}{c_a} - \sum_{\substack{y \text{ adj} \\ \text{to } a}} v_y \cdot P_{ay}\right]$$

$$= c_a\left[1 - \sum_{\substack{y \text{ adj} \\ \text{to } a}} v_y \cdot P_{ay}\right]$$

$v_y \cdot P_{ay}$ = Probability of reaching a before b for walk starting at y

$\sum_{\substack{y \text{ adj} \\ \text{to } a}} v_y \cdot P_{ay} =$ Probability of returning to a on walk from a to b (i.e. not escaping)

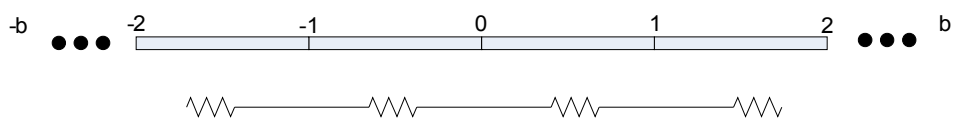
$\left[1 - \sum_{\substack{y \text{ adj} \\ \text{to } a}} v_y \cdot P_{ay}\right] =$ Probability of escape $= P_{escape}$

$\therefore\ i_a = c_a \cdot P_{escape}$

$i_a = v_a \cdot C_{eff} = C_{eff}$

Equating the two expressions we have: $c_a \cdot P_{escape} = C_{eff}$ or $P_{escape} = \dfrac{C_{eff}}{c_a}$

1 Dimension:



Replace all edges by 1Ω resistors.

$R_{eff} = \dfrac{k}{2}$ (we have 2 paths for $b = k$) and $C_{eff} = \dfrac{2}{k}$

Therefore, $\lim_{k \to \infty} C_{eff} = 0$. This means that if we start at the origin we will always come back, in the one dimensional case.

2 Dimensions:

To do this we will short all resistors in concentric squares in the lattice. We will thus determine a lower bound for R_{eff} and show that it is still infinite.

For the first few squares we have $R_{eff} = 1/4, 1/12, 1/20\ldots$

$R_{eff} \geq \dfrac{1}{4} + \dfrac{1}{12} + \dfrac{1}{20} + \cdots$

$R_{eff} \geq \dfrac{1}{4}\left[1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \cdots\right] \geq \dfrac{1}{4}\ln n$

This implies that $\lim_{n \to \infty} C_{eff} = 0$ and that $\lim_{n \to \infty} P_{escape} = 0$

3 Dimensions:

In the 3-dimensional case we can construct a tree rooted at the origin that branches. If we do this properly we get the sum as an upper bound:

$$R_{eff} \le \frac{1}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right] = \frac{1}{3}\left[\frac{1}{1 - \frac{2}{3}}\right] = 1$$

$\Rightarrow$ Probability of escape is some number between 0 and 1.