Random Walks on graphs

- Undirected
  - Lattice-graph made up of grid points connected by edges (special case)
- Directed
  - Associated with page rank

Definition:
A harmonic function on an undirected graph designates exterior and interior points. For exterior points, the value of the function is fixed. For interior points, the value is the weighted average of its adjacent neighbors. To obtain the values of interior points, we need to set up and solve a set of linear equations.

Claim:
A harmonic function takes on its max and min on the boundary.

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The difference of two harmonic functions is harmonic.

Suppose g, h are two harmonic functions with the same boundary conditions, then the harmonic function g-h on those points is zero. Since the function takes on its max and min on the boundary points, the interior points must also be zero. Therefore g and h must be the same function.

Claim: Given some boundary conditions, there is only one harmonic function.

Analogy between electrical circuits and random walks on graphs

Electrical circuit \( \leftrightarrow \) undirected graph
resistance \( r_{xy} \) \( \leftrightarrow \) at \( x \), prob of going to \( y \)
conductance \( c_{xy} = 1 / r_{xy} \)

\[ P_{xy} = c_{xy} / \sum c_{xy} = c_{xy} / c_x \]

Current Source

\[ V_s = 1 \]

Vb = 0

Electrical network

Current source induces a voltage on all nodes and current through all edges
Claim: voltage at x \(\leftrightarrow\) probability that a random walk starting at x reaches a before b

Proof:

For every edge x \(\rightarrow\) y

\[
\sum_y i_{xy} = 0 \quad \sum_y (v_x - v_y)c_{xy} = 0
\]

\[
\sum_y v_x c_{xy} = \sum_y v_y c_{xy}
\]

\[
v_x = \sum_y v_y \frac{c_{xy}}{c_x} = \sum_y v_y p_{xy}
\]

Voltage is therefore harmonic since \(V_x\) is the weighted sum of voltages of its neighbors.

Let P(x) be the probability that random walk started at x reaches a before b

\[
P(x) = \sum P(y) * P_{xy}
\]

P(x) is also a harmonic function, with the same weights as the previous function

By our claim above, the two functions must be the same. \(V(x) = P(x)\).

Fun Problem:

How many steps does it take to get distance n away from a starting point.

Need to solve equation

\[
T(i) = 1 + \frac{1}{2} * T(i-1) + \frac{1}{2} * T(i+1)
\]

\[
T(i) = -i^2
\]

To get a general solution, we need to solve

\[
T(i-1) - 2 * T(i) + T(i+1) = 0
\]

We get the general solution to be \(a + bi\)