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Growth Models without Preferential Attachment

We can grow graphs using the following method.

1. At each unit of time add a vertex
2. With probability ∂ select 2 vertices uniformly at random and draw an edge between them.

<u>What is the degree distribution in a graph grown in this way?</u>

Let $d_0(t)$ be the expected number of vertices of degree k at time t.

$$d_0(t+1) = d_0(t) + 1 - 2\partial\left(\frac{d_0(t)}{t}\right)$$

This sum is the number of isolated vertices at the previous time, plus the one just added, minus the expected number of isolated vertices to be attached to an edge at this step

$$d_k(t+1) = d_k(t) + 2\partial\left(\frac{d_{k-1}(t)}{t}\right) - 2\partial\left(\frac{d_k(t)}{t}\right)$$

This sum is the number of degree k vertices at the previous time, plus the expected number of degree k-1 vertices to have an edge added at this step, minus the expected number of degree k vertices to be attached to an edge at this step

Now we will consider solutions to $d_k(t)$ of the form $P_k t$.

From the above equations we get…

$$(t+1)P_0 = P_0 t + 1 - 2\partial(P_0 t / t)$$

$$\therefore P_0 = \frac{1}{1+2\partial}$$

$$(t+1)P_k = P_k * t + 2\partial P_{k-1} - 2\partial P_k$$

$$P_k = \frac{2\partial}{1+2\partial} P_{k-1} = \left(\frac{2\partial}{1+2\partial}\right)^k P_0$$

$$\therefore d_k(t) = \left(\frac{1}{1+2\partial}\right)\left(\frac{2\partial}{1+2\partial}\right)^k t$$

Components in the grown graph without preferential attachment

1. Goal: a generating function for component size of a component containing a randomly selected vertex.
    - *Note*: If a graph has two components, one of size 2, and one of size 4, and we chose a random component, the average component size would be 3. But we are choosing a vertex at random, so the average size is actually 4(4/6) + 2(2/6) = 10/3, not 3.

- Let $N_k(t)$ be the expected number of components of size k.
    - $N_1(t+1) = N_1(t) + 1 - 2\delta \dfrac{N_1(t)}{t}$
        - For each time step, we add one vertex (which will be an additional component of size 1), and then we choose two vertices at random, and add an edge with probability □, possibly removing a component of size 1.
    - $N_k(t+1) = N_k(t) + \delta \sum_{j=1}^{k-1} [\dfrac{jN_j(t)}{t} \dfrac{(k-j)N_{k-j}(t)}{t}]^{t} - 2\delta \dfrac{kN_k(t)}{t}$
        - For each time step, we can add components of size k by joining any two components with sizes j and k-j, and we can remove a component of size k if we attach an edge between it and another component.
- Consider solutions of the form $N_k(t) = a_k t$.
    - After a lot of arithmetic,
        - $a_1 = \dfrac{1}{1+2\delta}$
        - $a_k = \dfrac{\delta}{1+2k\delta} \sum_{j=1}^{k-1} j(k-j) a_j a_{k-j}$
    - a_k is proportional to the probability that a component selected at random has k elements.
        - $\sum_{k=0}^{\infty} N_k(t) = \sum_{k=0}^{\infty} a_k t \rightarrow \sum_{k=0}^{\infty} a_k = \dfrac{\sum_{k=0}^{\infty} N_k(t)}{t} < 1$
            - Formula sums over all components
        - $\sum_{k=0}^{\infty} k a_k = \sum_{k=0}^{\infty} \dfrac{k N_k(t)}{t} = 1$
            - Formula sums over all vertices

Generating function for component size of a component containing a randomly selected vertex

- Let $g(x)$ be the generating function for the distribution of component sizes where the coefficient of x^k is the probability that a vertex, chosen at random, is in a component of size k.
- We will derive: $g = -2\partial xg' + 2\partial xgg' + x$
- Rewrite equations for a_1 and a_k:
    - $(1+2\delta)a_1 = 1$
    - $(1+2k\delta)a_k = \delta \sum_{j=1}^{k-1} j(k-j) a_j a_{k-j}$
- Multiply kth equation by kx^k and sum:
    - $-x + \sum_{k=1}^{\infty} k a_k x^k + 2\delta x \sum_{k=1}^{\infty} a_k k^2 x^{k-1} = \delta \sum_{k=1}^{\infty} k x^k \sum_{j=1}^{k-1} j(k-j) a_j a_{k-j}$ (*)
        - The -x term on the left side occurs because the equation for a_1 is different from the

    other equations for other k.
  - The third term of the left side has an x taken out of the sum to yield a desired exponent of k-1 (that will come into play later)
  - *Note*:
    - $g(x)=\sum_{k=1}^{\infty} k a_k x^k$
    - $g'(x)=\sum_{k=1}^{\infty} k^2 a_k x^{k-1}$
- Transforming (*),
  - $-x+g(x)+2\delta x g'(x)=\delta x \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} j(k-j)(j+k-j)x^{k-1}a_j a_{k-j}$
    - Recognize the second and third terms of the left side as products involving g and g'.
    - Again, pull out an x from the right hand side to yield the desired exponent of x.
    - Rewrite k in the right side as $j + k - j$
  - $-x+g(x)+2\delta x g'(x)=\delta x \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} j^2(k-j)x^{k-1}a_j a_{k-j}+\delta x \sum_{k=1}^{\infty}\sum_{j=1}^{k-1} j(k-j)^2 x^{k-1}a_j a_{k-j}$
    - Note that the two terms on the right side are actually equivalent (this can be shown with a substitution of variables, if j and k-j are reversed.
    - Also note that each of the double-sums is equivalent to g' times g.
    - Thus, we get:
  - $-x+g+2\delta x g'=2\delta x g' g$
  - $g'=\frac{1}{2\delta}\frac{-x+g}{gx-x}=\frac{1}{2\delta}\frac{1-\frac{g}{x}}{1-g}$
- If $g(1) = 1$, then this function is indeterminate at this point.
- On Monday 2/26, we will show that if $g(1) < 1$, there will almost surely be a giant component.