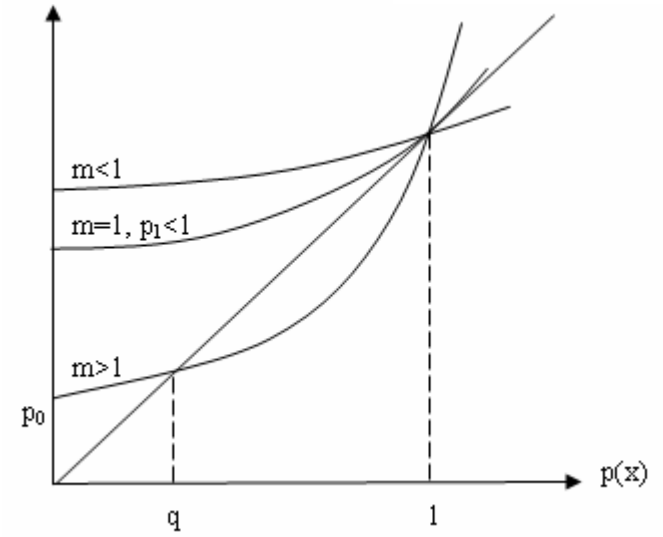
Expected Size of Distinct Families

$P_0, P_1, P_2, \ldots$

$$f(x) = \sum_{i=0}^{\infty} P_i x^i$$

$$f(f(x))$$

$$f_j(x) = f_{j-1}(f(x))$$



positive coefficients
=> curves are convex

$m = f'$

How many children are there in the jth generation?

$$\lim_{j\to\infty} f_j(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

If $a_1 = a_2 = \ldots = 0$, then $\lim_{j\to\infty} f_j(x) = q$. Thus, q is the probability it will die out.

Let Z_i be the random variable denoting the size of ith generation. Let q be the extinction probability.
Assume there were k children in the first generation. Then the extinct probability would be q^k.

$$P[Z_1 = k \mid extinction] = \frac{P[extinction \mid Z_1 = k]P[Z_1 = k]}{P[extinction]} = \frac{q^k P_k}{q} = q^{k-1} P_k$$

Expected size of Z_1 given extinction:

$$E[Z_1 \mid extinction] = \sum_{k=0}^{\infty} k q^{k-1} P_k = f'(q)$$

$$E[Z_1] = f'(1)$$

Expected size of ith generation given extinction:

$E[Z_i \mid extinction] = [f'(q)]^i$ with independence

Expected size of the tree = $\sum_{i=0}^{\infty} E[Z_i \mid extinction] = \sum_{i}^{\infty} [f'(q)]^i = \frac{1}{1-f'(q)}$, where $f'(q) < 1$

$f'(q) < 1$ when

1) m < 1, the curve crosses line $f(x) = x$ at $x = 1$

2) m > 1, the curve crosses line $f(x) = x$ at $x = 1 \,\&\, x = q$

Q: What happens when m = 1?

n fixed numbers

$$E(\sum_{i=0}^{\infty} x_i) = \sum_{i=0}^{\infty} E(x_i) = nE(x_1)$$

What if n was a random variable?

$$E(\sum_{i=0}^{\infty} x_i) = \Pr ob(n=1)E(x_1 \mid n=1) + \Pr ob(n=2)E(x_1 + x_2 \mid n=2) + \ldots$$

$$= \Pr ob(n=1)E(x_1) + \Pr ob(n=2)E(x_1 + x_2) + \ldots$$

$$= \Pr ob(n=1)E(x_1) + 2\Pr ob(n=2)E(x_1) + \ldots$$

Note: Because x_1 and x_2 are identically distributed, $E(x_1 + x_2) = 2E(x_1)$.

$$= E(x_1)E(n)$$
$$= E^2(x_1)$$

Generating Functions

a_0, a_1, … $\sum_{i=0}^{\infty} a_i x^i$

1, 1, 1, … $f(x) = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

1, 2, 3, … $f'(x) = 1 + 2x + 3x^2 + \ldots = \frac{1}{(1-x)^2}$

$xf'(x) = x + 2x^2 + 3x^3 + \ldots$

1, 4, 9, 16, … $xf''(x) + f'(x) = 1 + 4x + 9x^2 + 16x^3 + \ldots$

A can be selected 0 or 1 time.
B can be selected 0, 1, or 2 times.
C can be selected 0, 1, 2, or 3 times.

If only A's are selected: $1+x$

If only B's are selected: $1+x+x^2$

If only C's are selected: $1+x+x^2+x^3$

$$(1+x)(1+x+x^2)(1+x+x^2+x^3)=1+3x+5x^2+6x^3+5x^4+3x^5+x^6$$