**Expected Size of Distinct Families**

$$P_0, P_1, P_2, \ldots$$

$$f(x) = \sum_{j=0}^{\infty} P_j x^j$$

$$f(f(x))$$

$$f_j(x) = f_{j-1}(f(x))$$

**positive coefficients**

$$=>$$ curves are convex

$$m = f'$$

How many children are there in the $j^{th}$ generation?

$$\lim_{j \to \infty} f_j(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

If $a_1 = a_2 = \ldots = 0$, then $\lim_{j \to \infty} f_j(x) = q$. Thus, $q$ is the probability it will die out.

Let $Z_i$ be the random variable denoting the size of $i^{th}$ generation. Let $q$ be the extinction probability.

Assume there were $k$ children in the first generation. Then the extinct probability would be $q^k$.

$$P[Z_1 = k \mid \text{extinction}] = \frac{P[\text{extinction} \mid Z_1 = k]P[Z_1 = k]}{P[\text{extinction}]} = \frac{q^k P_k}{q} = q^{k-1} P_k$$

Expected size of $Z_1$ given extinction:

$$E[Z_1 \mid \text{extinction}] = \sum_{k=0}^{\infty} k q^{k-1} P_k = f'(q)$$

$$E[Z_1] = f'(1)$$

Expected size of $i^{th}$ generation given extinction:

$$E[Z_i \mid \text{extinction}] = [f'(q)]^i \text{ with independence}$$
Expected size of the tree = \( \sum_{i=0}^{\infty} E[Z_i \mid \text{extinction}] = \sum_{i} [f'(q)]^i = \frac{1}{1 - f'(q)} \), where \( f'(q) < 1 \)

\( f'(q) < 1 \) when

1) \( m < 1 \), the curve crosses line \( f(x) = x \) at \( x = 1 \)

2) \( m > 1 \), the curve crosses line \( f(x) = x \) at \( x = 1 \) \& \( x = q \)

Q: What happens when \( m = 1 \)?

n fixed numbers

\[ E(\sum_{i=0}^{\infty} x_i) = \sum_{i=0}^{\infty} E(x_i) = nE(x_i) \]

What if \( n \) was a random variable?

\[ E(\sum_{i=0}^{\infty} x_i) = \Pr \; \text{ob}(n = 1)E(x_i \mid n = 1) + \Pr \; \text{ob}(n = 2)E(x_i + x_2 \mid n = 2) + \ldots \]

\[ = \Pr \; \text{ob}(n = 1)E(x_i) + \Pr \; \text{ob}(n = 2)E(x_i + x_2) + \ldots \]

\[ = \Pr \; \text{ob}(n = 1)E(x_i) + 2 \Pr \; \text{ob}(n = 2)E(x_i) + \ldots \]

Note: Because \( x_1 \) and \( x_2 \) are identically distributed, \( E(x_i + x_2) = 2E(x_i) \).

\[ = E(x_i)E(n) = E^2(x_i) \]

Generating Functions

\[ a_0, \ a_1, \ldots \sum_{i=0}^{\infty} a_i x^i \]

1, 1, 1, ...

\[ f(x) = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} \]

1, 2, 3, ...

\[ f'(x) = 1 + 2x + 3x^2 + \ldots = \frac{1}{(1-x)^2} \]

\[ xf'(x) = x + 2x^2 + 3x^3 + \ldots \]

1, 4, 9, 16, ...

\[ xf''(x) + f'(x) = 1 + 4x + 9x^2 + 16x^3 + \ldots \]
A can be selected 0 or 1 time.
B can be selected 0, 1, or 2 times.
C can be selected 0, 1, 2, or 3 times.

If only A’s are selected: \[ 1 + x \]
If only B’s are selected: \[ 1 + x + x^2 \]
If only C’s are selected: \[ 1 + x + x^2 + x^3 \]

\[(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3) = 1 + 3x + 5x^2 + 6x^3 + 5x^4 + 3x^5 + x^6 \]