CS 485

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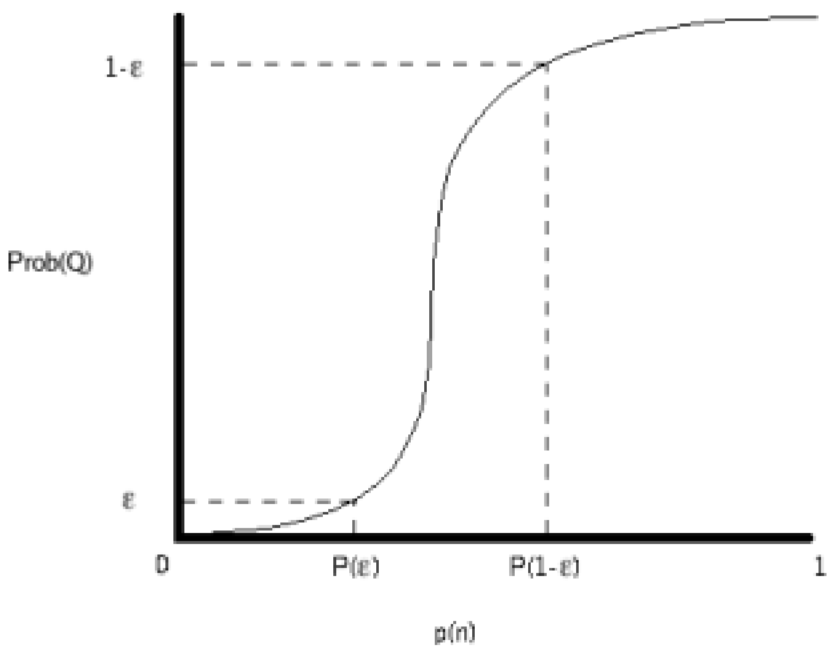
WTS: Every property of N_p has a threshold.

Let $p(\epsilon)$ be the function $p(n)$ such that the probability that $N_{p(n)}$ has a property Q is ϵ. We need to show that for any $\epsilon > 0$ that $\exists$ a constant m such that $p(1-\epsilon) \leq m \cdot p(\epsilon)$.



We never want $p(\epsilon) = \frac{1}{n^{2/3}}$ and $p(1-\epsilon) = \frac{log\ n}{n}$. We would like $p(\epsilon) = \frac{0.1}{n}$ and $p(1-\epsilon) = \frac{0.9}{n}$.
i.e. $p(\epsilon) \leq c \cdot p(\epsilon)$ where c is a constant.

Theorem: Every increasing property for N_p has a threshold.
Proof: Let $0 < \epsilon \leq 1/2$ and m be an integer such that $(1-\epsilon)^m \leq \epsilon$.
ex: if $\epsilon = 0.1$ and $(1-\epsilon) = 0.9$ then $0.9^{32} = 0.034 < 0.1$.

Consdier N_q - the union of m independent copies of $N_{p(\epsilon)}$. Since Q is an increasing property, if one or more of the m independent copies of $N_{p(\epsilon)}$ has the property Q, then N_q has the property Q. Thus, if N_q does not have property Q, then none of the $N_{p(\epsilon)}$ have the property Q.

$$Prob(N_q \notin Q) = Prob\left(\forall N_{p(\epsilon)}, N_{p(\epsilon)} \notin Q\right) = \left[1 - Prob\left(N_{p(\epsilon)} \in Q\right)\right]^m = (1-\epsilon)^m \leq \epsilon$$

$$[Prob(N_q \notin Q) \leq \epsilon] \Rightarrow [Prob(N_q \in Q) \geq (1-\epsilon)] \qquad (1)$$

The union is equivalent to N_q where $q = 1 - (1 - p(\epsilon))^m$ since:

$(1 - p(\epsilon))$ is the probability that an integer is not in a given copy.

$(1 - p(\epsilon))^m$ is the probability that an integer is not in any of the m copies.

$1 - (1 - p(\epsilon))^m$ is the probability that an integert is in one or more copies.

$$\begin{aligned}
q = 1 - (1 - p(\epsilon))^m &= 1 - (1 - m \cdot p(\epsilon) + \ldots) \leq m \cdot p(\epsilon) \\
Prob(N_q \in Q) &\leq Prob(N_{m \cdot p(\epsilon)} \in Q) \qquad\qquad (2) \\
(1 - \epsilon) &\leq Prob(N_q \in Q) \leq Prob(N_{m \cdot p(\epsilon)} \in Q) \qquad by\ (1)\ and\ (2) \\
p(1 - \epsilon) &\leq m \cdot p(\epsilon)
\end{aligned}$$

If $0 < \epsilon \leq 1/2$ then $p(\epsilon) \leq p(1/2) \leq p(1 - \epsilon) \leq m \cdot p(\epsilon)$. Asymptotically $p(\epsilon)$, $p(1/2)$ and $p(1 - \epsilon)$ are equivalent $\Rightarrow p(1/2)$ is a threshold.