CS485 Lecture 6 Jason Lui
2/3/06 Victor Kwok

**More on Second Moment Method**

Let x be a non-negative random variable. Then:

$$P(x=0) \le \frac{\sigma^2}{E^2(X)} = \frac{E(X^2)}{E^2(X)} - 1$$

The emergence of cycles in a graph G(n, p):

- Occurs when $p = \Theta(1/n)$ e.g. p = 1/1000n
- Doesn't occur if $\lim_{n->\infty} \frac{p(n)}{1/n} = 0$

Let x be the number of cycles in graph G:

$$E(x) = \sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2} p^k$$

$$E(x) \le \sum_{k=3}^{n} \frac{n(n-1)...(n-k)}{k!} \frac{(k-1)!}{2} p^k$$

$$E(x) \le \sum_{k=3}^{n} \frac{n^k}{2k} p^k \le \sum_{k=3}^{n} (np)^k$$

*What if p is asymptotically less than 1/n? (i.e.* $\lim_{n->\infty} np = 0$*)*

Consider: $\sum_{k=0}^{\infty} a^k = 1 + a + a^2 + ... = \frac{1}{1-a}$ , for all a <1

$$\Rightarrow E(x) \le \sum_{k=3}^{n} (np)^k = 0$$

because k starts at 3 which means it doesn't include the first term of the series 1. Therefore almost surely a graph selected at random has no cycle of p is asymptotically less than 1/n.

*What if np = constant, c?*

$$E(x) = \sum_{k=3}^{n} \binom{n}{k} \frac{(k-1)!}{2} p^k = \frac{1}{2} \sum_{k=3}^{n} \frac{n(n-1)...(n-k)}{kn^k} (np)^k$$

- If c < 1, it converges.
- If c >= 1, it diverges, but why?

Let us add up the first log n terms:

$$E(x) \geq \frac{1}{2} \sum_{k=3}^{n} \left( \frac{(n - \log n)}{n} \right)^k (np)^k \geq \frac{(n - \log n)}{n} \log n \to \log n \text{ as } n \to \infty$$

**Other Strcutures**

$N = \{1,2,\ldots,n\}$

Flip a coin which has head with probability p and put integer in the set the head occurs.
$N_p = \{1,2,5,9,13\}$

Does $N_p$ contain an arithmetic progression of length $k$?
$a, a+b, a+2b, a+3b,\ldots,a+(k-1)b$

Yes, arithmetic progression of length $k$ abruptly appears when $p$ reaches $n^{-\frac{2}{k}}$

Why?
There are $n^2$ potential numbers of arithmetic progression
Let $X_k$ be the expected number of arithmetic progression, then
$E(X_k) = n^2 p^k$

If $p << n^{-\frac{2}{k}}$
$E(X_k) << n^2 \times n^{-2} << 1$
$\therefore \lim_{n \to \infty} E(X_k) = 0$

If p >> $n^{-\frac{2}{k}}$
$\therefore \lim_{n \to \infty} E(X_k) = \infty$

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**Aside: Covariance**

$Cov(x,y) = E((x-E(x)(y-E(y))$

$Var(x+y) = E[((x+y)-E(x+y))^2]$
$= Var(x) + Var(y) + 2Cov(x,y)$

If x and y > 0, then
$Cov(x,y) < E(xy)$

We now want to establish that

$\lim_{n->\infty} E(X_k > 0) = 1$, for $p >> n^{-\frac{2}{k}}$

Let $I_i$ be the indicator variable for the $i^{th}$ arithmetic progression, then

$X_k = I_1 + I_2 + \ldots$
$Var(X_k) = \sum_i \sum_j Cov(I_i, I_j)$