Notes for CS 485
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1. Isolated verticies disapear at $p = \frac{\ln n}{n}$

Let the random variable X be the number of isolated verticies.
From before, we know that if
$p = \frac{c \ln n}{n}$ where c is a constant, then

$E(x) = n^{1-c}$

$\lim_{n\to\infty} E(x) = \lim_{n\to\infty} n^{1-c} = \begin{cases} 0 & c>1\text{, which implies } Prob(X=0)=1 \\ \infty & c<1\text{, which } \underline{\text{might}} \text{ imply } Prob(X=0)<1 \end{cases}$

For $E(x) \to \infty$, $Prob(X=0)$ could still approach 1, if all of the isolated verticies are confined to only a few graphs in the ensemble of possible graphs:

graph:	G_0	G_1	G_2	G_3 ...
X :	∞	0	0	0 ...

[Table shows all graphs, and # isolated verticies for each graph]

To rule this possibility out, we use the 2nd moment method to show that $\sigma(X)$ is much smaller than $E(x)$ as $n \to \infty$

2. Proof of Markov and Chebyshev Inequalities.

Let X be a random variable s.t. $X \geq 0$.
The Markov inequality states that for all $a>0$
$Prob(X \geq a) \leq \frac{E(x)}{a}$ or by setting $b = \frac{a}{E(x)}$, $Prob(X \geq bE(x)) \leq \frac{1}{b}$
Proof: Let $p(x)$ be probability distribution on X. Then

$$E(x) = \int_0^\infty x p(x) dx = \int_0^a x p(x) dx + \int_a^\infty x p(x) dx \geq 0 + \int_a^\infty x p(x) dx \geq \int_a^\infty a p(x) dx$$

$E(x) \geq a \int_a^\infty p(x)dx = a\left[Prob(X \geq a)\right]$, and finally

$Prob(X \geq a) \leq \frac{E(x)}{a}$.

2. Continued:

The Chebychev Inequality states that for any distribution over the random variable X, with mean $E(X)$ and variance σ^2, and any $t \geq 0$, $\text{Prob}(|X - E(x)| \geq \sigma t) \leq \frac{1}{t^2}$

Proof: Square both sides in probability and let $X' = (x - E(x))^2$, $a' = \sigma^2 t^2$

$$\text{Prob}(|X - E(x)| \geq \sigma t) = \text{Prob}((X - E(x))^2 \geq \sigma^2 t^2) = \text{Prob}(x' \geq a')$$

$X' \geq 0$ so we can use Markov's inequality

$$\text{Prob}(x' \geq a') \leq \frac{E(x')}{a'} = \frac{E((X - E(x))^2)}{\sigma^2 t^2} = \frac{\sigma^2}{\sigma^2 t^2}$$ by the definition of variance

$$\text{Prob}(|X - E(x)| \geq \sigma t) \leq \frac{\sigma^2}{\sigma^2 t^2} = \frac{1}{t^2}$$

3. The Second Moment Method.

Consider a random variable X s.t. $X \geq 0$ and the mean $E(x)$ and the deviation σ are known

$$\text{Prob}(X = 0) = \text{Prob}(E(x) - X = E(x)) \leq \text{Prob}(|X - E(x)| \geq E(x))$$

Let $E(x) = t\sigma$, then

$$\text{Prob}(x = 0) \leq \text{Prob}(|X - E(x)| \geq t\sigma) \leq \frac{1}{t^2} = \frac{\sigma^2}{E^2(x)}$$

$$\sigma^2 = E((X - E(x))^2) = E(x^2 - 2x E(x) + E^2(x)) = E(x^2) - 2E(x)E(x) + E^2(x)$$
$$= E(x^2) - E^2(x) \quad \text{so that}$$

$$\text{Prob}(X = 0) \leq \frac{E(x^2) - E^2(x)}{E^2(x)} = \frac{E(x^2)}{E^2(x)} - 1$$

Suppose that the distribution on X, depends on a variable n.

If $\lim_{n \to \infty} \frac{E(x^2)}{E^2(x)} = 1$, then $\lim_{n \to \infty} \text{Prob}(X = 0) = 0$

3. continued

The 2nd moment method applied to isolated verticies

Let X = # isolated verticies.

$X = \sum_{i=1}^{n} X_i$ where $X_i = \begin{cases} 1 & \text{if ith vertex is isolated} \\ 0 & \text{otherwise} \end{cases}$

$$X^2 = \sum_i \sum_j X_i X_j = \sum_i X_i^2 + \sum_{i \neq j} X_i X_j$$

$\sum_{i=1}^{n} X_i^2 = \sum_{i=1}^{n} X_i = X$ since $X_i^2 = X_i$

$$E(X^2) = E\left(\sum_{i=1}^{n} X_i^2\right) + E\left(\sum_{i \neq j} X_i X_j\right) = E(X) + \sum_{i \neq j} E(X_i X_j)$$

$= E(x) + n(n-1) E(X_1 X_2)$ since $E(X_i X_j)$ is the same for all $i, j, i \neq j$

$$E(X_1 X_2) = \text{Prob}(X_1 X_2 = 1) = \text{Prob}(X_1 = 1)\,\text{Prob}(X_2 = 1 \mid X_1 = 1)$$

$\text{Prob}(X_1 = 1) = (1-p)^{n-1}$ since there can be no edges to the other $n-1$ verticies

$\text{Prob}(X_2 = 1 \mid X_1 = 1) = (1-p)^{n-2}$ since we need not consider vertex 1.

$E(X) = n(1-p)^{n-1}$ and $E(X_1 X_2) = (1-p)^{2n-3}$

$E(x^2) = E(x) + n(n-1)E = n(1-p)^{n-1} + n(n-1)(1-p)^{2n-3}$

$$\frac{E(x^2)}{E^2(x)} = \frac{1}{n(1-p)^{n-1}} + \frac{n(n-1)(1-p)^{2n-3}}{n^2(1-p)^{2n-2}} = \frac{1}{n(1-p)^{n-1}} + \frac{n-1}{n}\,\frac{1}{1-p}$$

For $p = \frac{c \ln n}{n}$, $n \to \infty$ and $c < 1$

$$\frac{E(x^2)}{E^2(x)} = \frac{1}{n^{1-c}} + \frac{1}{1 - \frac{c \ln n}{n}} = 0 + \frac{1}{1-0} = 1$$

Therefore, for $c < 1$, $\lim_{n \to \infty} \text{Prob}(X = 0) = 0$

so that for $p < \frac{\ln n}{n}$, there will almost certainly be an isolated vertex.