COM S 485
Lecture 5 Notes
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Second Moment Method

Disappearance of Isolated Vertices

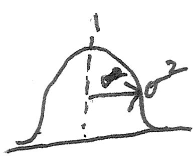
As p increases, there is a transition from collections of trees, to cycles, to the emergence of a giant component. As p increases further, isolated vertices are swept up by the giant component.

Let x be the number of isolated vertices.

$p = c(\ln n)/n$ $\qquad$ $E(x) = n^{1-c}$

The limit of E(x), as n goes to ∞, is 0 if $c \geq 1$ and 1 if $c \leq 1$. $\qquad$ Prob(x=0) = 1

*There is the possibility that, given a series of graphs, for the first graph $x = \infty$ while for the remaining graphs $x = 0$. The Second Moment Method will be used to rule out this possibility.



*if σ^2 is small, we should not have the above situation ($x = \infty$ for one graph, $x = 0$ for the others).

*First, a bit about Markov's Inequality and Chebyshev's Inequaltiy:

Theorem: Markov's Inequality

Let x be a random variable that is nonnegative. Then, $P(x \geq a) \leq E(x)/a$

$$E(x)=\int_0^\infty xP(x)dx = \int_0^a xP(x)dx + \int_a^\infty xP(x)dx$$

-if we throw the first integral away, we get a lower bound on E(x): $E(x) \geq \int_a^\infty xP(x)dx$

-x is always greater than a, so: $\int_a^\infty xP(x)dx \geq a\underbrace{\int_a^\infty xP(x)dx}_{P(x\geq a)}$

$P(x \geq a) \leq E(x)/a$

-replace a by bE(x): $P(x \geq bE(x)) \leq E(x)/bE(x) = 1/b$

$$\boxed{P(x \geq bE(x)) \leq 1/b}$$

*We can get a tighter bound by taking into account the variance….

Theorem: Chebyshev's Inequality:

Let x be a random variable. Then the probability that $[|x-E(x)| \geq t\sigma] < 1/t^2$

*With Chebyshev's Inequality, we measure distance in terms of the standard deviation.

Proof: $Prob[|x - \mu| \geq t\sigma] = Prob[(x - \mu)^2 \geq t^2\sigma^2]$

Note: will sometimes use μ or m in place of E(x)

Apply Markov's Inequality:

$Prob[(x - \mu)^2 \geq t^2\sigma^2] \leq E[(x - \mu)^2]/t^2\sigma^2 = \sigma^2/ t^2\sigma^2 = 1/t^2$ ($\sigma^2 = E[(x- \mu)^2]$)

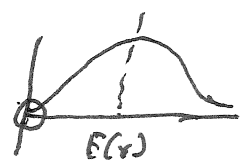
- $t^2\sigma^2$ is what was 'a' back in Markov's Inequality

Second Moment Method

$Prob[|x - E(x)| \geq E(x)] \leq \sigma^2/E^2(x)$ ($t = E(x)/ \sigma$ and replace)

Now clearly for a nonnegative random variable x:

$Prob(x = 0) \leq Prob[|x - E(x)| \geq E(x)] \leq \sigma^2/E^2(x)$



*We want to prove that P(x = 0), so need to show that $\sigma^2/E^2(x)$ goes to 0 (i.e. variance goes to 0 with respect to the expected value)

Under what conditions does $\sigma^2/E^2(x)$ go to 0?

$\sigma^2 = E[(x-E(x))^2] = E(x^2) - 2E(x)E(x) + E^2(x) = E(x^2) - E^2(x)$

$Prob[|x - E(x)| \geq E(x)] \leq \sigma^2/E^2(x) = [E(x^2) - E^2(x)]/E^2(x) = E(x^2)/E^2(x) - 1$

If $E(x^2)$ is asymptotically less than or equal to $E^2(x)$, then Prob(x = 0) goes to 0

Back to the disappearance of isolated vertices: applying the Second Moment Method:

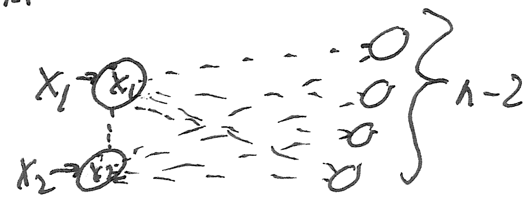
x= number of isolated vertices

$Prob\ (x = 0) \longrightarrow E(x^2) \leq E^2(x)$

-need to calculate value of x^2

$x = \underbrace{x_1 + x_2 + \ldots\ldots + x_n}_{\text{indicator variables}}$ $x_i = 1$ if the vertex is isolated, 0 otherwise

$$E(x^2) = \sum_{i=1}^{n} E(x_i^2) + \sum_{i \neq j} (x_i x_j) = \sum_{i=1}^{n} E(x_i) + n(n-1)E(x_1 x_2) = E(x) + n(n-1)(1-p)^{2(n-1)-1}$$



Probability that x_1 is isolated = $(1-p)^{n-1}$
Probability that x_2 is isolated is the same as above, except don't count the edge between x_1 and x_2 twice

Now $E(x^2)/E^2(x) = 1/E(x) + [n(n-1)/n^2][(1-p)^{2(n-1)-1}/(1-p)^{2(n-1)}]$

$E(x^2)/E^2(x) = 1/E(x) + 1/(1-p) = 1/n(1-p)^{n-1} + 1/(1-p)$ $\quad\quad (E(x) = n(1-p)^{n-1})$

For $p = c(\ln n)/n$ and $c < 1$: $E(x)$ goes to ∞

$E(x^2)/E^2(x) = 1/n^{1-c} + 1/(1-(c(\ln n)/n)) = 1$

Thus, by the Second Moment Method, can claim that $\text{Prob}(x = 0)$ goes to 0