CS 485 Lecture 4 - Monday 30 January 2006

Scribe for 1st half: Jeffrey M Wong (jmw92)

Note: possible midterm date: Fri 7 April.

Structure of Graphs of $G(n,p)$ as p increases

$p(n) = 0 \;\ldots\; \frac{1}{n^2} \;\ldots\; \frac{1}{n^{3/2}} \;\ldots\; \frac{1}{n\log n} \;\ldots\; \frac{1}{n} \;\ldots\; \frac{1}{2} \;\; \frac{3}{4} \;\; 1$

1) Forest of trees (no cycles) while $p = o(\frac{1}{n})$ [i.e. $\frac{p(n)}{1/n} \to 0$ as $n \to \infty$]
2) Cycles will appear as soon as $p = \Theta(\frac{1}{n})$ [i.e. $\frac{p(n)}{1/n} \to \text{const}$]
   - each component is a tree or unicyclic
   - no component exists of size larger than $\log n$
3) At $p = \frac{1}{n}$, an abrupt phase transition occurs in which a giant component appears. This phase transition occurs as a double jump whereby:
   - At $p < \frac{1}{n}$: largest component has size $\sim \log n$
   - At $p = \frac{1}{n}$: largest component has size $\sim n^{2/3}$
   - At $p > \frac{1}{n}$: largest component has size $\sim \text{const} \cdot n$
4) As p increases larger components get swallowed up into the giant component. At $p = \frac{1}{4}\frac{\log n}{n}$: There is only the giant component plus isolated vertices.
5) At $p = \frac{\log n}{n}$: connected graph (no isolated vertices left)
6) At $p = \text{const}$: graph has diameter 2.

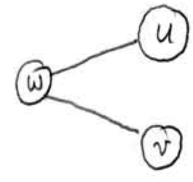
More on phase transition at $p = \frac{1}{n}$:

- This is a "sharp" transition, implying the existence of a "threshold" at $p = \frac{1}{n}$.
- We say that $f(n)$ is a threshold for a property if
  - for all $f_1(n)$ such that $\frac{f_1(n)}{f(n)} \to 0$, almost no graph has the property, and
  - for all $f_2(n)$ such that $\frac{f(n)}{f_2(n)} \to 0$, almost every graph has the property.

- Thresholds occur for all properties of $G(n,p)$ that we list.
- Thresholds occur for other structures besides $G(n,p)$.

Review from Prev. Lecture: Graph has diameter at most 2 when $p = \text{const}$.

- Define an unordered pair of vertices (u,v) to be "bad" if no vertex w exists that is adjacent to both u and v.



- Let x = # of unordered pairs in graph that are bad.
- Let x_i = indicator variable for ith pair $(1 < i < \binom{n}{2})$:
  $x_i = 1$ if bad pair, $x_i = 0$ if good pair
- Then $x = \sum x_i$ and expected value $E(x) = \sum E(x_i)$
  But all $E(x_i)$ are the same, so $E(x) = \binom{n}{2} E(x_1)$
- For a given u and v, what is the probability that there does not exist a vertex w adjacent to both u and v?
  - Prob. that a given vertex is not adjacent to both is $(1-p^2)$
  - There are $(n-2)$ possible vertices, so the probability that no w exists is $(1-p^2)^{n-2}$
- So $E(x) = \binom{n}{2}(1-p^2)^{n-2} = \frac{n(n-1)}{2} c^{n-2}$ where c is a constant less than 1.
  As $n \to \infty$, $E(x) \to 0$. Therefore, for a random graph, the expected number of bad pairs is zero. Thus, the graph has diameter at most 2.
- What if $E(x)$ had been 1? We <u>cannot</u> conclude that all (or many) graphs do have bad pairs, because they need not be uniformly distributed.
- This proof relies on $p = \text{const}$ because $(1-p^2)^{n-2} \to 0$ only if p is a constant.
  For example, $(1-\frac{d}{n})^n \to e^{-d} \neq 0$.

CS 485 Lecture 4, part B

Disappearance of isolated vertices.

Let x be the number of isolater vertices

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        E(x) = n(1-p)^(n-1)

        (1-p)^n = e^(ln(1-p)^n)

        =  e^(n ln(1-p))

                [ ln(1-p) = -p - (p^2)/2 - (p^3)/3 - ... ]

        =  e^(-n(p + (p^2)/2 + (p^3)/3 + ...))

        =  e^(-np) * e^(-n((p^2)/2 + (p^3)/3 + ...))

        =  e^(-np) * e^(-n(p^2)(1/2 + p/3 + (p^2)/4 + ...))

                [ p = (c ln n) / n ]

        =  e^(-c ln n) * e^((-n(c^2)((ln n)^2))/(n^2))*(1/2 + (c ln n)/3n + ...)

lim     -> e^(-c ln n)

E(x) = n(n^-c) = n^(1-c)

c<1, E(x) -> inf
c>1, E(x) -> 0

*note:  p = (c ln n)/n
        (1 - (c ln n)/n)^n = e^(c ln n) = n^-c ??
        (1 - d/n)^n = e^-d
        We have only proved this for constant d so this is not correct.
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