CS 485 - Lecture 35

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Recall:
Markov's inequality: $Prob(x \geq am) \leq \frac{1}{a}$
Chebyshev's inequality: $Prob(|x - m| \geq a\sigma) \leq \frac{1}{a^2}$

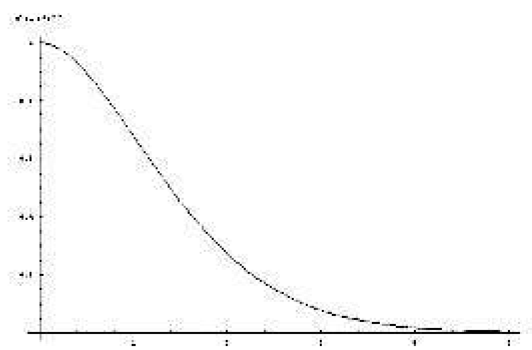
Chernoff bounds:

Let $x_1, x_2, ..., x_n$ be independent random variables from distribution:

$$x_i = \begin{cases} 1 \ with \ probability \ p \\ 0 \ with \ probability \ \ (1-p) \end{cases}$$

Consider $s = \sum_{i=1}^{n} x_i$. $E(x_i) = p$. Define $m = E(s) = E(\sum x_i) = np$.

Theorem: For any $\delta > 0$, $\quad Prob\,[s > (1+\delta)m] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{m}$.



Proof: For any $\lambda > 0$

$$\begin{aligned} Prob\,[s > (1+\delta)m] &= Prob\left[e^{\lambda s} > e^{\lambda(1+\delta)m}\right] \\ &\leq \frac{E(e^{\lambda s})}{e^{\lambda(1+\delta)m}} \quad by \ Markov's \ inequality \end{aligned}$$

First, cosider the numerator:

$$\begin{aligned} E(e^{\lambda s}) &= E(e^{\lambda \sum x_i}) = E(\prod e^{\lambda x_i}) = \prod \ E(e^{\lambda x_i}) \\ &= \prod (e^{\lambda} p + 1 - p) = \prod \left[(e^{\lambda} - 1)p + 1\right] \\ e^{x} = 1 + x + \frac{x^2}{2} + ... \ &\Rightarrow \ e^{\lambda} \geq 1 + x \\ Let \ \ x = p(e^{\lambda} - 1) \ \ then & \\ E(e^{\lambda s}) &\leq \prod e^{p(e^{\lambda}-1)} \end{aligned}$$

Set $\lambda = ln(1+\delta)$.

$$\begin{aligned} Prob\,[s > (1+\delta)m] &\leq \frac{\prod e^{p(e^{\lambda}-1)}}{e^{\lambda(1+\delta)m}} = \frac{\prod e^{p\delta}}{(1+\delta)^{1+\delta}m} \\ &= \frac{e^{pn\delta}}{(1+\delta)^{(1+\delta)m}} = \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{m} \end{aligned}$$

Therefore, $Prob\,[s > (1+\delta)m] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{m}$. Simplify bound:

- very small δ

$$\begin{aligned} ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \\ (1+x)ln(1+x) &= x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} + ... \\ ln(1+\delta)^{1+\delta} &= (1+\delta)ln(1+\delta) \approx \delta + \frac{\delta^2}{2} - \frac{\delta^3}{6} + ... \\ (1+\delta)^{1+\delta} &= e^{\delta + \frac{\delta^2}{2} - \frac{\delta^3}{6} + ...} \approx e^{-\frac{\delta^2}{2}} \qquad for\ small\ \delta \end{aligned}$$

- large δ

$$\begin{aligned} Prob\,[s \geq (1+\delta)m] &\leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{m} \\ &\leq \left[\frac{e}{1+\delta}\right]^{(1+\delta)m} \\ &\leq \left(\frac{1}{2}\right)^{(1+\delta)m} \qquad for\ \delta \geq 2e-1 \end{aligned}$$

What about $p(x) = \frac{1}{\pi}\frac{1}{1+x^2}$? The mean for this funciton can be defined as $m = a \overset{lim}{\rightarrow} \infty \int_{-a}^{a} xp(x)dx$. However, this function does not have a variance.