CS 485 – Lecture 33– April 17, 2006

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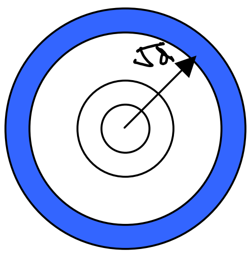
Review: High Dimensional Data

For high dimensions, almost all the volume of the unit cube is outside the unit sphere.

How far apart can 2 points be on a:

1) sphere: 2
2) cube: $2\sqrt{d}$

All the probability will be found in the shaded area (the annulus).

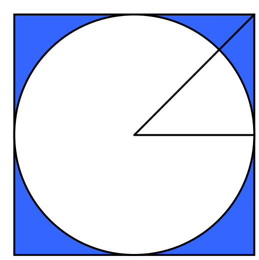


Suppose points are placed at random and we want to calculate the distance between points.

$$x = (x_1, x_2, \ldots, x_d)$$
$$y = (y_1, y_2, \ldots, y_d)$$
$$dist^2(x, y) = \sum_{i=1}^{d} (x_i - y_i)^2$$

Deviation of sum of random variables from expected value of sum

$$\Pr ob\left(\left| \sum_{i=1}^{n} x_i - E(\sum_{i=1}^{n} x_i \right| \geq c \right) \leq e^{-\frac{2c^2}{\sigma^2}}$$



Another Problem:

How do you generate points at random on the surface of a sphere?

Possible Solution:

In 2-dimensions, you might try to generate points uniformly on a square (i.e. rand function in Matlab).

Solution:

Discard all points outside of circle and project remaining points onto surface.

Why does this method not work in high dimensions?

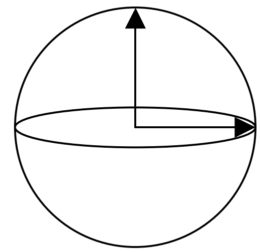
Most of the area of a hypercube will lie outside the sphere.

Generate points according to the following distribution:

$(x_1, x_2, ..., x_d)$ $\qquad e^{-\frac{x_1^2 + x_2^2 + ... + x_d^2}{2}} = e^{-\frac{r^2}{2}}$

Then normalize the points:

$$\frac{x_1}{\sqrt{x_1^2 + x_2^2 + ... + x_d^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2 + ... + x_d^2}}, ..., \frac{x_d}{\sqrt{x_1^2 + x_2^2 + ... + x_d^2}}$$



<u>Now we want to know:</u>

Generate two points on unit sphere.
After generating first point, rotate coordinate to place it on North Pole.
Generate 2nd point.

$$dist^2 = (1 - \frac{x_1}{\sqrt{x_1^2 + x_2^2 + ... + x_d^2}})^2 + (\frac{x_2^2}{x_1^2 + x_2^2 + ... + x_d^2}) + ... + (\frac{x_d^2}{x_1^2 + x_2^2 + ... + x_d^2})$$

$$= 1 - \frac{2x_1}{\sqrt{x_1^2 + x_2^2 + ... + x_d^2}} + (\frac{x_1^2}{x_1^2 + x_2^2 + ... + x_d^2}) + (\frac{x_2^2}{x_1^2 + x_2^2 + ... + x_d^2}) + ... + (\frac{x_2^2}{x_1^2 + x_2^2 + ... + x_d^2})$$

$$= 2 - \frac{2x_1}{\sqrt{x_1^2 + x_2^2 + ... + x_d^2}} = 2 - 2x_1$$

<u>Clarification Question: Why did we rotate in this manner?</u>

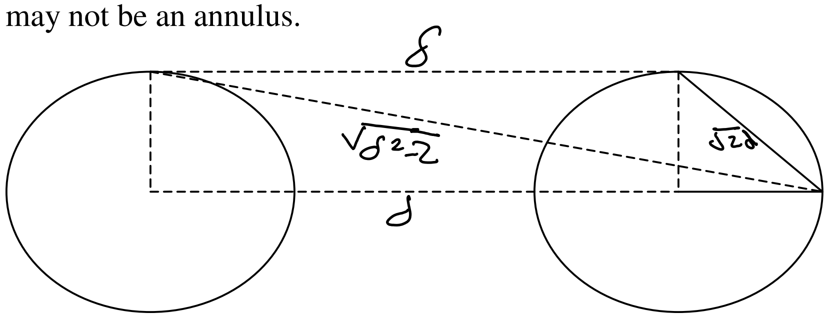
We wanted to make things easier to calculate the distance; we simply rotated the coordinate system.

$E(dist^2) = 2$

$E(dist) = \sqrt{2}$

Points on average are perpendicular.

Gaussian distribution used here. Depending on what distribution is used, there may or may not be an annulus.



Two Gaussians, points would be on 2 annuluses.
Pick 2 random points and calculate the distance between them.

Let δ = distance between vectors

Distance between points = $\sqrt{d^2 - 2}$

<u>Question: What if spheres of radius $\sqrt{d}$?</u>

If two points generated by some Gaussian, they will be $\sqrt{2d}$ distance apart.

If two points generated by different Gaussians, they will be $\sqrt{\delta^2 + 2d}$

To determine which Gaussian generated, calculate all pairwise distances and compare distance to $\sqrt{2d}$ or $\sqrt{\delta^2 + 2d}$

If $\sqrt{\delta^2 + 2d} \geq \sqrt{2d} + c$

$$\sqrt{\delta^2 + 2d} = \sqrt{2d}\left(\sqrt{2 + \frac{\delta^2}{4d}}\right) = \sqrt{2d}\,(1 + \frac{\delta^2}{4d} + ...) \geq \sqrt{2d} - c$$

$$\sqrt{2d}\,(\frac{\delta^2}{4d}) \geq c$$

$$\delta^2 \geq \frac{c * 4d}{\sqrt{2}\sqrt{d}} \geq \frac{4c}{\sqrt{2}}\sqrt{d}$$

$$\delta \geq c' * d^{1/4}$$

But we can do better than this.