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Today we looked at high-dimensional data, trying to create an intuition that would work for it. Most of us currently imagine things in 2 or 3-dimensions, but things often change as dimensionality increases.

1 High Dimensions

The volume of a cube increases as the number of dimensions increases.

The volume of a sphere, on the other hand, does not necessarily grow as the number of dimensions increases. As a matter of fact, the volume increases until 11 dimensions and then starts decreasing, approaching a volume of zero as the number of dimensions approaches infinity!

2 Volume of Sphere

We can represent the volume of a sphere in d dimensions as:

$$\int_{x_1=-1}^{1} \int_{x_2=-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \cdots \int_{x_d=-\sqrt{1-x_1^2-x_2^2-\ldots-x_{d-1}^2}}^{\sqrt{1-x_1^2-x_2^2-\ldots-x_{d-1}^2}} dx_1\, dx_2\, \ldots\, dx_d$$

We can transform this into polar coordinates as follows:

$$V(d) = \int\!\!\int \!\ldots\! \int_0^1 r d\theta_1\, r d\theta_2 \ldots r d\theta_{r-1} dr = \int d\Omega \int_0^1 r^{d-1}\, dr$$

We know that $\int d\Omega$ is just the area of d, so taking the integral we get:

$$V(d) = A(d) \frac{rd}{d}\bigg|_0^1 = \frac{A(d)}{d}$$

We can make a table to show how area relates to volume:

d	A(d)	V(d)
2	2π	π
3	4π	$\frac{4}{3}\pi$
$\vdots$	$\vdots$	$\vdots$

Now let us define $I(d)$:

$$I(d) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2+\ldots+x_d^2)}\, dx_1\, dx_2 \ldots dx_d = \left[\int_{-\infty}^{\infty} e^{-x_1^2}\, dx_1\right] = \left(\sqrt{\pi}\right)^d$$

$$\begin{aligned} I(d) &= \int d\Omega \int_0^{\infty} e^{-r^2} r^{d-1}\, dr \\ &= A(d) \int_0^{\infty} e^{-t} \frac{dt}{2\sqrt{t}} \left(\sqrt{t}\right)^{d-1} \\ &= \frac{1}{2} A(d) \int_0^{\infty} e^{-t} t^{\frac{d}{2}-1}\, dt \\ &= \frac{1}{2} A(d)\, \Gamma\!\left(\frac{d}{2}\right) \end{aligned}$$

Where

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

We can therefore see that:

$$A(d) = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

and:

$$V(d) = \frac{2}{d} \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$