Today we looked at high-dimensional data, trying to create an intuition that would work for it. Most of us currently imagine things in 2 or 3-dimensions, but things often change as dimensionality increases.

1 High Dimensions

The volume of a cube increases as the number of dimensions increases.

The volume of a sphere, on the other hand, does not necessarily grow as the number of dimensions increases. As a matter of fact, the volume increases until 11 dimensions and then starts decreasing, approaching a volume of zero as the number of dimensions approaches infinity!

2 Volume of Sphere

We can represent the volume of a sphere in d dimensions as:

\[ V(d) = \int_{x_1=-1}^{1} \int_{x_2=-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} \cdots \int_{x_d=-\sqrt{1-x_1^2-x_2^2-\cdots-x_{d-1}^2}}^{\sqrt{1-x_1^2-x_2^2-\cdots-x_{d-1}^2}} dx_1 \, dx_2 \cdots dx_d \]

We can transform this into polar coordinates as follows:

\[ V(d) = \int d\Omega \int_{0}^{\infty} r \, dr \cdots \int_{0}^{\infty} r \, dr \cdots \int_{0}^{\infty} r \, dr = \int d\Omega \int_{0}^{1} r^{d-1} \, dr \]

We know that \( \int d\Omega \) is just the area of d, so taking the integral we get:

\[ V(d) = A(d) \frac{r}{d} \bigg|_{0}^{1} = \frac{A(d)}{d} \]

We can make a table to show how area relates to volume:

<table>
<thead>
<tr>
<th>( d )</th>
<th>( A(d) )</th>
<th>( V(d) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( 2\pi )</td>
<td>( \pi )</td>
</tr>
<tr>
<td>3</td>
<td>( 4\pi )</td>
<td>( \frac{4}{3} \pi )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Now let us define \( I(d) \):

\[ I(d) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1^2+\cdots+x_d^2)} \, dx_1 \, dx_2 \cdots dx_d = \left[ \int_{-\infty}^{\infty} e^{-x_1^2} \, dx_1 \right] = (\sqrt{\pi})^d \]

\[ I(d) = \int d\Omega \int_{0}^{\infty} e^{-t^2} \, r^{d-1} \, dr \]

\[ = A(d) \int_{0}^{\infty} e^{-t} \, dt \frac{2}{2\sqrt{d}} (\sqrt{\pi})^{d-1} \]

\[ = \frac{1}{2} A(d) \int_{0}^{\infty} e^{-t} t^{d-1} \, dt \]

\[ = \frac{1}{2} A(d) \Gamma\left( \frac{d}{2} \right) \]

Where
\[ \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt \]

We can therefore see that:

\[ A(d) = \frac{2\pi^\frac{d}{2}}{\Gamma(\frac{d}{2})} \]

and:

\[ V(d) = \frac{2}{d} \frac{\pi^\frac{d}{2}}{\Gamma(\frac{d}{2})} \]