CS 485 Lecture 3

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Structure of Graphs as *p* Increases

Given a graph G(n, p), we note that p is a function of n. e.g. $p = \frac{1}{\log n}$, $\frac{1}{n^2}$, etc.

The structure of a large graph undergoes several phase transitions as p is increased:

Phase Transitions	Observations
1. $p = \frac{1}{n^2}$, $\frac{1}{n^{3/2}}$, $\frac{1}{n \log n}$	▪ p is of order o($1/n$). ▪ Graphs have a number of components which are all trees (no cycles) and have size ≤ log n.
2. $p = \frac{1}{10n}$, $\frac{1}{2n}$	▪ p is of order $\Theta(1/n)$. ▪ Cycles begin to appear in the graph. All components are either trees or unicyclic with size ≤ log n.
3. $p = \frac{1}{n}$, $\frac{2}{n}$	▪ If p < $1/n$, components have size ≤ log n. ▪ If $p = 1/n$, a giant component of size $n^{2/3}$ appears. ▪ If $p > 1/n$, a giant component has size cn, where c is a constant. ▪ As p increases further, the giant component absorbs all other smaller components.
4. $p = \frac{\log n}{4n}$	▪ Graph contains only a giant component plus isolated vertices. All smaller components absorbed.
5. $p = \frac{\log n}{n}$	▪ Graph is connected (all isolated vertices absorbed).
6. $p = \frac{1}{10}$, $\frac{1}{3}$, $\frac{1}{2}$	▪ p is a constant. ▪ Graph almost surely has a diameter of 2.

Theorem: Given G(n, p) with p a constant independent of n, then the graph almost surely has a diameter of 2.

Proof: Pick any 2 vertices from the graph. Call them node 1 and 2:

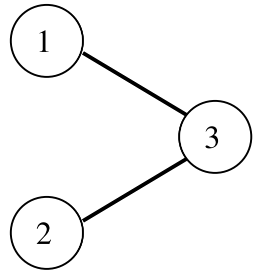

Figure 1

We wish to find the probability finding a third node (node 3 in Fig.1) that is adjacent to both node 1 and 2.

Let x be the number of unordered pairs of vertices (u, v) such that there is no other vertex w adjacent to both u and v (we call such a (u, v) pair a bad pair). If there are no bad pairs, then the graph has diameter 2. In other words, if $E(x) \rightarrow 0$ as $n \rightarrow \infty$, then for large n, there are almost no graphs that are not of diameter 2.

(Note that the converse, i.e. if $E(x) \rightarrow \infty$ as $n \rightarrow \infty$, then all graphs will almost surely have bad pairs, is not true. Suppose we have graphs G_1, G_2, G_3, G_4, etc, and $x_1 \rightarrow \infty$, $x_2 \rightarrow 0$, $x_3 \rightarrow 0$, $x_4 \rightarrow 0$, etc as $n \rightarrow \infty$, then $E(x) \rightarrow \infty$ even though every other graph except G_1 has no bad pairs)

Next, number pairs of vertices from 1 to $N = \binom{n}{2}$. Let x_i be indicator variables where:

$$x_i = \begin{cases} 0 & \text{if i}^{\text{th}} \text{ pair of vertices is not bad} \\ 1 & \text{if i}^{\text{th}} \text{ pair of vertices is bad} \end{cases}$$

x then is just $x_1 + x_2 + x_3 + \ldots + x_N$ and by linearity $E(x)$ is simply $\sum_{i=1}^{N} \mathrm{x_i}$. Furthermore, since the expected value of each x_i are equal, $E(x)$ can be simplified to $\binom{n}{2}E(x_1)$.

To find $E(x_1)$, we refer back to Fig.1. What is the probability that both node 1 and 2 is connected to node 3? p^2. So the probability that both node 1 and 2 is not connected to node 3 is $(1 - p^2)$. Finally, the probability that both node 1 and node 2 is not connected to all the other n – 2 vertices (including node 3) is $(1 - p^2)^{n-2}$. But this is just $E(x_1)$.

$$\text{So, } E(x) = \binom{n}{2}E(x_1) \tag{1}$$

$$= \binom{n}{2}(1 - p^2)^{n-2} \tag{2}$$

If $(1 - p^2)$ is a constant < 1, $E(x) \rightarrow 0$ as $n \rightarrow \infty$.