Lecture 25 Soo Yeon Lee
March 27, 2006 Jang Ho Kim

- Announcements
  - Send references to bhoward@cs.cornell.edu

- Random walks on expanders
  - Real world examples: Finding random websites
    Comparing search engines
  - Regular degree d graphs

    $P = (P_1, P_2, \ldots, P_n)^T$ where P_i is the probability of being at vertex i

    Intially, $P = (1, 0, \ldots, 0)^T$ ← start at vertex 1

    $$A \cdot \frac{P}{d} \to P' \quad (P' \text{ is the new prob})$$

    Suggest normalizing A by dividing by d: $\frac{A}{d}$

    Stationary probability: $\left(\frac{1}{n}, \cdots, \frac{1}{n}\right) = \frac{u}{n}$ where $u = (1, \cdots, 1)$

    Theorem: Let P be an arbitrary probability distribution.

    Then $\left|\left(\frac{A}{d}\right)^S P - \frac{u}{n}\right|_1 \le \sqrt{n}\left(\frac{\lambda}{d}\right)^S$ where $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$

    and s is the number of steps.

    Proof: Since $\frac{u}{n}$ is an eigenvector of $\frac{A}{d}$ with eigenvalue 1,

    $$\frac{A}{d} \cdot \frac{u}{n} = \frac{u}{n} \quad \text{and} \quad \left(\frac{A}{d}\right)^S \cdot \frac{u}{n} = \frac{u}{n}$$

    So, $$\left(\frac{A}{d}\right)^S P - \frac{u}{n} = \left(\frac{A}{d}\right)^S P - \left(\frac{A}{d}\right)^S \frac{u}{n} = \left(\frac{A}{d}\right)^S \left(P - \frac{u}{n}\right)$$

    $$\left|\left(\frac{A}{d}\right)^S P - \frac{u}{n}\right|_2 = \left|\left(\frac{A}{d}\right)^S \left(P - \frac{u}{n}\right)\right|_2$$

    The coordinates of P sum to 1 as do the coordinates of $\frac{u}{n}$. Thus the coordinates of $\left(P - \frac{u}{n}\right)$ sum to zero. Therefore, $\left(P - \frac{u}{n}\right)$ is orthogonal to u and hence is in the subspace spanned by the remaining eigenvectors $v_2, v_3, \ldots, v_n$.

$$\left|\left(\frac{\mathrm{A}}{\mathrm{d}}\right)^{S}\left(\mathrm{P}-\frac{u}{n}\right)\right|_2 \le \left(\frac{\lambda_2(\mathrm{A})}{\mathrm{d}}\right)^{S}\left|\mathrm{P}-\frac{u}{n}\right|_2$$

$$\boxed{\begin{aligned} p-\frac{u}{n} &= \left|(p_1,\ldots,p_n)-(\tfrac{1}{n},\ldots,\tfrac{1}{n})\right|_2 \le \left|(1,0,\ldots,0)-(\tfrac{1}{n},\ldots,\tfrac{1}{n})\right|_2 \\ &= (1-\frac{1}{n})+\frac{1}{n}+\ldots+\frac{1}{n} \\ &= 1-\frac{2}{n}+\frac{1}{n}+\frac{n-1}{n^2} = 1-\frac{2}{n}+\frac{1}{n} \end{aligned}}$$

Since $\left|p-\frac{u}{n}\right|_2 \le 1$

$$\left|\left(\frac{\mathrm{A}}{\mathrm{d}}\right)^{S}\left(\mathrm{P}-\frac{\mu}{n}\right)\right|_2 \le \left(\lambda\left(\frac{\mathrm{A}}{\mathrm{d}}\right)\right)^{S}$$

To convert this to a one-norm, use $|v|_1 \le \sqrt{n}|v|_2$

Thus $\left|\left(\frac{\mathrm{A}}{\mathrm{d}}\right)^{S}\left(\mathrm{P}-\frac{\mu}{n}\right)\right|_1 \le \sqrt{n}\left(\lambda\left(\frac{\mathrm{A}}{\mathrm{d}}\right)\right)^{S} = \sqrt{n}\left(\frac{\lambda(\mathrm{A})}{\mathrm{d}}\right)^{S}$

<u>Corollary</u>: $\left|\left(\frac{\mathrm{A}}{\mathrm{d}}\right)^{S}\left(\mathrm{P}-\frac{\mu}{n}\right)\right|_1 \le \sqrt{n}\left(1-\frac{h^2(G)}{2\mathrm{d}^2}\right)^{S}$

Where $h(G) = \frac{\min\limits_{|S|\le\frac{n}{2}} |\partial S|}{|S|}$ * $|\partial S|$ is the number of edges out of S

$\Delta(G) = \lambda_1 - \lambda_2$ * Recall $h^2(G) \le 2d\Delta(G)$

<u>Proof</u>: $\lambda\left(\frac{A}{d}\right) \ge \lambda_2\left(\frac{A}{d}\right) = \frac{\lambda_2(A)}{d} = \frac{\lambda_1-\Delta}{d} = 1-\frac{\Delta}{d}$

$$\left|\left(\frac{\mathrm{A}}{\mathrm{d}}\right)^{S}\left(\mathrm{P}-\frac{\mu}{n}\right)\right|_1 \le \sqrt{n}\left(1-\frac{\Delta}{\mathrm{d}}\right)^{S} \qquad \boxed{h^2(G)\le 2d\Delta(G) \Rightarrow \frac{h^2(G)}{2d} \le \Delta(G)}$$

$$\therefore \left|\left(\frac{\mathrm{A}}{\mathrm{d}}\right)^{S}\left(\mathrm{P}-\frac{\mu}{n}\right)\right|_1 \le \sqrt{n}\left(1-\frac{h^2(G)}{2\mathrm{d}^2}\right)^{S}$$