CS 485 lecture notes
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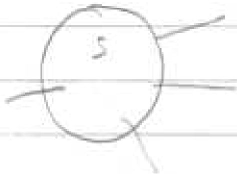

$$h(G) = \min_{|S| \le n/2} \frac{|\partial S|}{|S|} \quad \text{expansion parameter}$$

$$\Delta G = \lambda_1 - \lambda_2 \quad \text{eigen gaps}$$

Dealing with regular degree d graph.

Theorem: $\frac{\Delta(G)}{2} \le h(G) \le \sqrt{2d\,\Delta(G)}$

Proof: $\lambda_1 = d$ (reg. deg. d graph)

To compute eigenvalue we need to determine λ_2

$\mu = (1, 1, \ldots, 1)$

$$\lambda_2 = \max_{x \perp \mu} \frac{x^T A x}{x^T x}$$

$tr(x) = 0$.

let S and T be disjoint subset of vertices

let μ_S and μ_T be a characteristic vector of sets S and T

$$\mu_S^T A \mu_T = |\text{edge}(S,T)|$$

let $v = \frac{\mu_S}{|S|} - \frac{\mu_{\bar{S}}}{|\bar{S}|}$ $\qquad$ trace $(v) = 0$

$$v^T v = \frac{|S|}{|S|^2} + \frac{|\bar{S}|}{|\bar{S}|^2} = \frac{1}{|S|} + \frac{1}{|\bar{S}|}$$

$$\left[(\mu_S - \mu_{\bar{S}}) A (\mu_S - \mu_{\bar{S}}) = E(S,S) + E(\bar{S},\bar{S}) - 2E(S,\bar{S}) \right.$$

$\left.\text{where } E(S,T) \text{ is the edge between vertex set } S \text{ and } T \right]$

$$v^T A v = \frac{2}{|S|^2} E(S,S) + \frac{2}{|\bar{S}|^2} E(\bar{S},\bar{S}) - 2\frac{1}{|S||\bar{S}|} E(S,\bar{S})$$

$E(S,S) + E(S,\bar{S}) = d|S|$ $\qquad$ since each vertex has degree d.

$E(S,S) = d|S| - E(S,\bar{S})$

$E(\bar{S},\bar{S}) = d|\bar{S}| - E(S,\bar{S})$

$$v^T A v = \frac{1}{|S|^2}\left[d|S| - E(S,\bar{S})\right] + \frac{1}{|\bar{S}|^2}\left[d|\bar{S}| - E(S,\bar{S})\right] - \frac{2}{|S||\bar{S}|} E(S,\bar{S})$$

$$= \cancel{2d}\, d\left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right) - \left[\frac{1}{|S|^2} + \frac{2}{|S||\bar{S}|} + \frac{1}{|\bar{S}|^2}\right] E(S,\bar{S})$$

$$= d\left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right) - \left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right)^2 E(S,\bar{S}).$$

$$\lambda_2 \geq \frac{v^T A v}{v^T v} = d - \left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right) E(S,\bar{S})$$

Choose S so that $E(S,\bar{S}) = h(G)|S|$ and $|S| \leq n/2$

$h(G)$ is $\min\limits_{|S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|}$

$h(G) = \frac{|\partial S|}{|S|} = \frac{E(S,\bar{S})}{|S|}$

$E(S,\bar{S}) = |S|\, h(G)$

$$\lambda_2 \geq d - \left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right)|S|\, h(G)$$

$$= d - \left(1 + \frac{|S|}{|\bar{S}|}\right) h(G)$$

$\lambda_2 \geq d - 2h(G)$

since $|S| \leq n/2$

$\Delta G = \lambda_1 - \lambda_2 \leq d - d + 2h(G)$

$\Delta G \leq 2h(G)$

$\frac{\Delta G}{2} \leq h(G)$