Lecture Notes 3/17/06
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Definition: Eigengap $\Delta(G) = \lambda_1 - \lambda_2$

Theorem: $\frac{\Delta(G)}{2} \leq h(G) \leq \sqrt{2\text{d}\,\Delta(G)}$
Pf:

Let S and T be disjoint subsets of vertices in G.

Let u_s and u_t be the characteristic vectors of S and T.

Observe that:

$u_S{}^T A u_T = |E(S,T)|$ where E(x,y) is the number of edges between vertices in x and y

Note that this is because the ith component of Au_T is the number of edges between vertices in T and vertex i. So $u_S{}^T A u_T$ is the sum of edges between vertices in T and vertices in S.

Let v = $\frac{u_S}{|S|} - \frac{u_{\bar{S}}}{|\bar{S}|}$

$$v^T v = \frac{|S|}{|S|^2} + \frac{|\bar{S}|}{|\bar{S}|^2} \quad \text{since} \quad u_S \cdot u_{\bar{S}} = 0$$

$$v^T v = \frac{1}{|S|} + \frac{1}{|\bar{S}|} \quad \text{[Eq 1]}$$

$$\begin{aligned} v^T A v &= v^T \left(A\frac{u_S}{|S|} - A\frac{u_{\bar{S}}}{|\bar{S}|}\right) = \left(\frac{u_S}{|S|}^T - \frac{u_{\bar{S}}}{|\bar{S}|}^T\right)\left(A\frac{u_S}{|S|} - A\frac{u_{\bar{S}}}{|\bar{S}|}\right) \\ &= \frac{1}{|S|^2}(u_S{}^T A u_S) + \frac{1}{|\bar{S}|^2}(u_{\bar{S}}{}^T A u_{\bar{S}}) - \frac{1}{|S||\bar{S}|}(u_S{}^T A u_{\bar{S}}) - \frac{1}{|S||\bar{S}|}(u_{\bar{S}}{}^T A u_S) \\ &= \frac{1}{|S|^2} E(S,S) + \frac{1}{|\bar{S}|^2} E(\bar{S},\bar{S}) - \frac{2}{|S||\bar{S}|} E(S,\bar{S}) \quad \text{(See observation above).} \end{aligned}$$

Since $E(S,S) + E(S,\bar{S}) = d|S|$ and $E(\bar{S},\bar{S}) + E(S,\bar{S}) = d|\bar{S}|$:

$$\begin{aligned} v^T A v &= \frac{1}{|S|^2}(d|S| - E(S,\bar{S})) + \frac{1}{|\bar{S}|^2}(d|\bar{S}| - E(S,\bar{S})) - \frac{2}{|S||\bar{S}|} E(S,\bar{S}) \\ &= d\left(\frac{1}{|S|} + \frac{1}{|\bar{S}|}\right) - \left(\frac{1}{|S|^2} + \frac{1}{|\bar{S}|^2} + \frac{2}{|S||\bar{S}|}\right) E(S,\bar{S}) \end{aligned}$$

$$= \quad d\left(\frac{1}{|S|}+\frac{1}{|\overline{S}|}\right)-\left(\frac{1}{|S|}+\frac{1}{|\overline{S}|}\right)^2 E(S,\overline{S}) \quad \text{[Eq 2]}$$

Combining Equations 1 and 2, we have:

$$\frac{v^T A v}{v^T v}=d-\left(\frac{1}{|S|}+\frac{1}{|\overline{S}|}\right)E(S,\overline{S})$$

Since $\lambda_2 = max_{x \perp u}\frac{x^T A x}{x^T x}$, $\lambda_2 \geq \frac{v^T A v}{v^T v}=d-\left(\frac{1}{|S|}+\frac{1}{|\overline{S}|}\right)E(S,\overline{S})$ [Eq 3]

If we select S such that $E(S,\overline{S})=h(G)|S|$ and $|S|\leq\frac{n}{2}$

$h(G)=min_{|T|\leq\frac{n}{2}}\frac{|\delta T|}{|T|}$ (see previous lecture)

Let S be the set T minimizes h(G)

h(G) = $\frac{|\delta S|}{|S|}$ = $\frac{E(S,\overline{S})}{|S|}$ Thus, $|S|h(G)=E(S,\overline{S})$ [Eq 4]

Thus,

$\lambda_2 \geq d-\left(\frac{1}{|S|}+\frac{1}{|\overline{S}|}\right)|S|h(G)$ [by Eq 3 and Eq 4]

$$\lambda_2 \geq d-\left(1+\frac{|S|}{|\overline{S}|}\right)h(G)$$

Since S is at most $n/2$ $\frac{|S|}{|\overline{S}|}$ is at most 1, therefore,

$$\lambda_2 \geq d-2h(G)$$

$$\Delta(G)=\lambda_1-\lambda_2 \leq d-(d-2\mathrm{h}(G))$$

$$\Delta(G)\leq 2\mathrm{h}(G)$$

$\Delta\frac{(G)}{2}\leq h(G)$ (as desired)

A similar proof exists for $h(G) \leq \sqrt{2\mathrm{d}\Delta(G)}$