We start by looking at the vertex importance process that we ended with last class. We saw that by iterating the redistribution process and weighting over and over, we approach the degree distribution. (Without weighting the process goes to infinity)

$$
\left[\begin{array}{c}
\text { Adjacency }  \tag{1}\\
\text { Matrix } \\
\mathbf{A}
\end{array}\right]\left[\begin{array}{c}
\text { Prob. } \\
\text { Dist. } \\
\mathbf{x}
\end{array}\right]=\left[\begin{array}{c}
\text { New } \\
\text { Prob. } \\
\text { Dist. }
\end{array}\right]
$$

Which gives us the equation $(\mathbf{A}-\lambda \mathbf{I}) \mathbf{x}=0$. This equation has a non-trivial solution iff $\operatorname{det}(\mathbf{A}-\lambda \mathbf{I})=0$. This will yield a polynomial in $\lambda$ of degree $n$, which has $n$ unique roots (the eigenvalues).

$$
\begin{aligned}
\mathbf{x} & =\sum_{i=1}^{n} c_{i} v_{i} \\
\mathbf{A x} & =\sum_{i=1}^{n} \lambda_{i} c_{i} v_{i}
\end{aligned}
$$

The difference between $\lambda_{1}$ and $\lambda_{2}$ determines how fast the importance process converges to a stable solution. Now, instead of looking at importances, lets instead to a random walk on the graph in which we pick each outgoing edge with equal probability. The node we start from begins with a probability of 1 , and get distributed to the nodes it's connected to on the first step.


Now instead of simply multiplying $x$ by the adjacency matrix $A$, we first multiply $x$ by the inverse of the degree distribution.

$$
\begin{aligned}
A D^{-1} x & =x & & \text { AD }{ }^{-1} \text { is not a symmetric matrix } \\
y & =D^{-1 / 2} x & & \text { variable substitution }\{y / x\} \\
x & =D^{1 / 2} y & & \text { solve for } x \\
A D^{-1} D^{1 / 2} y & =D^{1 / 2} y & & \text { substitute into original } \\
D^{-1 / 2} A D^{-1 / 2} y & =y & & \text { now we have a symmetric matrix }
\end{aligned}
$$

Notice that y is not actually the vector of stationary probabilities, x is. After solving for y , we need to get x by computing $D^{1 / 2} y$. So what does $D^{-1 / 2} A D^{-1 / 2}$ look like?

$$
D=\left[\begin{array}{ccc}
d_{1} & \ldots & 0  \tag{2}\\
0 & \ldots & 0 \\
0 & \ldots & d_{n}
\end{array}\right] D^{-1}=\left[\begin{array}{ccc}
1 / d_{1} & \ldots & 0 \\
0 & \ldots & 0 \\
0 & \ldots & 1 / d_{n}
\end{array}\right] D^{-1 / 2}=\left[\begin{array}{ccc}
\sqrt{1 / d_{1}} & \ldots & 0 \\
0 & \ldots & 0 \\
0 & \cdots & \sqrt{1 / d_{n}}
\end{array}\right]
$$

$$
\begin{gather*}
D^{-1 / 2} A D^{-1 / 2} y=\left[\begin{array}{c}
\text { Elements }: \\
\frac{a_{i j}}{\sqrt{d_{i} d_{j}}}
\end{array}\right]\left[\begin{array}{c}
\sqrt{d_{1}} \\
\ldots \\
\sqrt{d_{n}}
\end{array}\right]=\left[\begin{array}{c}
\text { the ith row is equal to : } \\
\left.\frac{a_{i 1} \sqrt{d_{1}}}{\sqrt{d_{i} d_{1}}}+\ldots+\begin{array}{c}
\frac{a_{i n} \sqrt{d_{n}}}{\sqrt{d_{i} d_{n}}}= \\
\frac{1}{\sqrt{d_{i}}}\left(\sum_{j=1}^{n} a_{i j}\right)=\frac{d_{i}}{\sqrt{d_{i}}}=\sqrt{d_{i}}
\end{array}\right]=\left[\begin{array}{c}
\sqrt{d_{1}} \\
\ldots \\
\sqrt{d_{n}}
\end{array}\right] \\
x=D^{1 / 2} y=\left[\begin{array}{ccc}
\sqrt{d_{1}} & \ldots & 0 \\
0 & \cdots & 0 \\
0 & \cdots & \sqrt{d_{n}}
\end{array}\right]\left[\begin{array}{c}
\sqrt{d_{1}} \\
\ldots \\
\sqrt{d_{n}}
\end{array}\right]=\left[\begin{array}{c}
d_{1} \\
\ldots \\
d_{n}
\end{array}\right]
\end{array}, .\right. \tag{3}
\end{gather*}
$$

We can find the first eigenvector by relaxation, the repeated process seen above. We know that the values used to find x start and stay positive, so this must be the first eigenvector.

For regular degree d graphs:

1. G is connected iff $\lambda_{1}>\lambda_{2}$
2. G is bipartite iff $\lambda_{1}=-\lambda_{n}$

Lemma: Consider a random connected regular degree d graph G. Let A be the adjacency matrix of this graph. Then,

$$
d=\lambda_{1}>\lambda_{2} \geq \lambda_{3} \geq \ldots \geq-d \quad \text { (last equality becomes strict if graph is bipartite) }
$$

Proof: Let $u=\left[\begin{array}{c}1 \\ \ldots \\ 1\end{array}\right]$ (note that this is an eigenvector). Let x be an eigenvector not proportional to $u$. Let $\max$ be the value of the maximum coordinate of x . Construct a set $\mathrm{S}=\left\{i \mid x_{i}=\max \right\}$. Since G is connected, there exists a vertex in $S$ connected to a vertex not in $S$. Call the vertex in $S$, $j$.
Consider multiplying $x$ times the $j^{\text {th }}$ row of $A$. Since the $j^{\text {th }}$ row has exactly $d 1$ 's and one of these corresponds to a component of $x$ less than $\max$, the product is less than or equal to $d * \max$.
Since $x$ is an eigenvector of $A$, the product of $x$ with the $j^{\text {th }}$ row of $A$ must be $\lambda x_{j}$. Therefore, if we have an odd cycle we can combine both of the results:

$$
\lambda x_{j}<d * \max \text { and } x_{j}=\max :
$$

Therefore,

$$
\lambda<d
$$

and

$$
\left.\left.\lambda_{1}=d\right\rangle \lambda_{2} \geq \lambda_{3} \geq \ldots \lambda_{n}\right\rangle-d
$$

Lemma: If the graph has exactly $k$ connected components then:

$$
\lambda_{1}=\lambda_{2}=\lambda_{3}=\ldots=\lambda_{k}>\lambda_{k+1}>\ldots
$$

