G(n,p) as we have seen is defined to be a graph with n vertices, each with possible edge appearing with probability p.

Interestingly, we see that almost all vertices are of degree within  $[(1 - \epsilon)m, (1 + \epsilon)m]$ . It is easy to see that

G(n, 1/2)

has average degree n/2 while

G(n, d/n)

has average degree d. We can also see that if we allow self loops, the probability of a vertex being of degree k is

$$Prob(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

If we sum over all the different possible k's then we see that it should add up to one (in this case, q = 1 - p therefore p + q = p + 1 - p = 1)

$$(p+q)^n = \sum_{k=0}^n \binom{n}{k} p^k q^{n-k}$$

If we take the derivative we get

$$n(p+q)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} p^{k} q^{n-k}$$

Which we see is just m, the mean. We can derive the last formula to get the variance:

$$n(n-1)(p+q)^{n-2} = \sum_{k=0}^{n} k(k-1) \binom{n}{p} p^{k-2} q^{n-k}$$

We can re-write this as

$$p^{2}n(n-1) = \sum_{k=0}^{n} k(k-1) \binom{n}{k} p^{k} (1-p)^{n-k}$$
$$= \sum_{k=0}^{n} k^{2} \binom{n}{k} p^{k} (1-p)^{n-k} - \sum_{k=0}^{n} k \binom{n}{k} p^{k} (1-p)^{n-k}$$

We know that the second part of the last equation is just m which equals np. We can re-write this as:

$$\sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} = p^2 n^2 - p^2 n + np$$

Therefore,

$$\begin{aligned} \sigma^2 &= \sum_{k=0}^n (np-k)^2 \binom{n}{k} p^k (1-p)^{n-k} \\ &= (np)^2 \underbrace{\sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}}_{\text{probability of } k=1} - 2np \underbrace{\sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k}}_{np} \\ &+ \underbrace{\sum_{k=0}^n k^2 \binom{n}{k} p^k (1-p)^{n-k}}_{np} \\ &= (np)^2 - 2(np)^2 + (np)^2 - np^2 + np \\ &= np - p^2n \\ &= np(1-p) \end{aligned}$$

We move on to the question: what is the highest degree vertex we expect in G(n, 1/2)? We expect a node of degree k with probability:

$$\lim_{n \to \infty} {n \choose k} p^k (1-p)^{n-k}$$

$$= \lim_{n \to \infty} \frac{n(n-1)(n-2)...(n-k)}{k!} \left(\frac{d}{n}\right)^k \left(1-\frac{d}{n}\right)^{n-k}$$

$$\approx \lim_{n \to \infty} \frac{n^k}{k!} * \frac{d^k}{n^k} e^{-d}$$

$$= \frac{d^k}{k!} e^{-d}$$

Lets try using  $d = \log n / \log \log n$  and play around with it a little:

$$\log d^{a}$$

$$= d \log d$$

$$= \frac{\log n}{\log \log n} [\log \log n - \log \log \log n]$$

$$= \log n$$

Since  $\log \log \log n$  is so close to 0. Therefore

$$d^d \approx n$$

Therefore

$$P(d) = \frac{1}{en}$$

There are n vertices, so the probability that some vertex is of degree  $\log n / \log \log n$  is 1/e = 0.36

We can now ask: how many triangles will we find in G(n, d/n)?

We expect there to be  $\binom{n}{3}$  triples of vertices each with a probability of  $\left(\frac{d}{n}\right)^3$  of being a triangle. Therefore:

$$\binom{n}{3}\left(\frac{d}{n}\right)^3 = \frac{n(n-1)(n-2)}{6}\frac{d^3}{n^3} = \frac{d^3}{6}$$

We can therefore see that the number of triangles we expect to find in the graph is independent of the number of nodes in the graph! We can find the expected number of triangles as well. Number all triples of vertices from 1 to  $\binom{n}{3}$  and we define:

Let 
$$I_i = \begin{cases} 0 & \text{if } i^{th} \text{ triple not a triangle} \\ 1 & \text{if } i^{th} \text{ is a triangle} \end{cases}$$

Therefore, the expected number of triangles is:

$$E(\sum I_i) = \sum E(I_i) = \binom{n}{3} \left(\frac{d}{n}\right)^3$$