CS 485 - Lecture 17

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Growth Model

At each unit of time we add a vertex, with probability δ add an edge connecting two vertices uniformly at random. We are wondering what happens to this model.
Let $N_k(t)$ be the expected number of components of size k at time t. So,

$$N_1(t+1) = N_1(t) + 1 - 2\delta \frac{N_1(t)}{t}$$

$$N_k(t+1) = N_k(t) + \delta \sum_{j=1}^{k-1} \frac{jN_j(t)}{t} \times \frac{(k-j)N_{k-j}(t)}{t} - \frac{2\delta k N_k(t)}{t}$$

Consider solution of form $N_k(t) = a_k t$. Plugging this value to the previous two equations, we get

$$a_1 = \frac{1}{1+2\delta} \qquad\qquad a_k = \frac{\delta}{1+2k\delta} \times \sum_{j=1}^{k-1} j(k-j)a_j a_{k-j}$$

a_k is not the probability, it is the constant of the proportionality. Notice that $a_k = \frac{N_k(t)}{t}$. If we sum up all the a_k, we are not going to get 1. So, to get 1, we have to multiply a_k by k so that

$$\sum_{k=0}^{\infty} k a_k = \sum_{k=0}^{\infty} \frac{kN_k(t)}{t} = 1$$

Notice if we take components of size k, multiply by the number of vertices in them, sum them up, we will get all the vertices in the graph. So, if we divide by t it is 1 (as shown above).

So, now we are going to construct a generating function for size of component containing a randomly selected vertex. So,

$$g(x) = \sum_{k=0}^{\infty} k a_k x^k$$

Now, we will try to find out what $g(x)$ is for the growing graph, because from $g(x)$ we can get all kinds of information. So, we will derive $g(x)$ satisfying the following equations:

$$g = -2\delta x g' + 2\delta x g g' + x$$

$$g' = \frac{1}{2\delta} \times \frac{1 - \frac{g(x)}{x}}{1 - g(x)}$$

The equation $a_1 = \frac{1}{1+2\delta}$ can be rewritten as

$$a_1 + 2\delta a_1 - 1 = 0$$

And we can rewrite the other equation as:

$$a_k + 2k\delta a_k = \delta \sum_{j=1}^{k-1} j(k-j) a_j a_{k-j}$$

Multiplying k^{th} term by kx^k and summing them up we get

$$-x + a_1 x + 2\delta a_1 x + \sum_{k=2}^{\infty} k a_k x^k + 2\delta \sum_{k=2}^{\infty} k^2 a_k x^k = \delta \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} j(k-j) k a_j a_{k-j} x^k$$

$$\implies -x + \underbrace{\sum_{k=1}^{\infty} k a_k x^k}_{g(x)} + 2\delta x \underbrace{\sum_{k=1}^{\infty} k^2 a_k x^{k-1}}_{g'(x)} = \underbrace{\delta \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} j(k-j) k x^k a_j a_{k-j}}_{2\delta x g'(x) g(x)}$$

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RHS: replace k with j+k-j to break equation into two sums: 1) j ,2) k-j

$$= \delta x \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} j(k-j)(j+k-j)\, x^{k-1} a_j a_{k-j}$$

$$= \delta x \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} j^2 (k-j)\, x^{k-1} a_j a_{k-j} + \delta x \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} j(k-j)^2\, x^{k-1} a_j a_{k-j}$$

let j→k-j , k-j→j. Then both sums become the same

$$= \delta x \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} j^2 (k-j)\, x^{k-1} a_j a_{k-j} + \delta x \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} (k-j) j^2\, x^{k-1} a_j a_{k-j}$$

Combine the sums

$$= 2\delta x \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} j^2 (k-j)\, x^{k-1} a_j a_{k-j}$$

By reordering, we can see that there is a convolution of g'(x) and g(x)

$$= 2\delta x \sum_{k=1}^{\infty} \sum_{j=1}^{k-1} j^2 a_j x^{j-1} (k-j)\, a_{k-j} x^{k-j}$$

$$g'(x) = \sum_{j=1}^{\infty} j^2 a_j x^{j-1} \qquad g'(x) = \sum_{i=1}^{\infty} i a_i x^i \quad \text{where i=k-j}$$

$$-x + g(x) + 2\delta g'(x) = 2\delta x g'(x) g(x)$$

$$g'(2\delta x - 2\delta x g) = x - g$$

$$\Rightarrow \quad g' = \frac{1}{2\delta} \frac{x-g}{x-xg} = \frac{1}{2\delta} \frac{1 - \frac{g(x)}{x}}{1-g(x)}$$

$g(x) = \sum_{i=1}^{\infty} i a_i x^i$ $\quad$ ia_i = fraction of vertices contained in a component of size i

g(1) = fraction of vertices contained in a finite component

If a giant component does not exist, g(1)≠1

$$g'(1) = \frac{1}{2\delta} \frac{1 - \frac{g(1)}{1}}{1-g(1)} = \frac{1}{2\delta} \frac{1-g(1)}{1-g(1)} = \frac{1}{2\delta}$$

$g'(x) = \sum_{i=1}^{\infty} i^2 a_i x^{i-1}$ = average size of components

$g'(x)$ = average size of finite components

$g'(1)$ = size of giant component

What if a giant component does not exist?
Then g(1)=1 and g'(1) becomes indeterminate
Use L'Hopital's Rule

$$g' = \frac{1}{2\delta}\frac{1-\frac{g(x)}{x}}{1-g(x)} = \frac{1}{2\delta}\frac{-\frac{g'(x)x-g(x)}{x^2}}{-g'(x)} = \frac{1}{2\delta}\frac{g'(x)x-g(x)}{x^2 g'(x)} = \frac{1}{2\delta}\frac{g'(1)-1}{g'(1)}$$

$\Rightarrow \qquad 2\delta(g')^2 - g' + 1 = 0$

$$g'^2 - \frac{1}{2\delta}g' + \frac{1}{2\delta} = 0$$ Use quadratic formula

$$g' = \frac{\frac{1}{2\delta} \pm \sqrt{\frac{1}{4\delta^2} - 4(1)(\frac{1}{2\delta})}}{2} = \frac{1}{4\delta} \pm \frac{1}{4\delta}\sqrt{1-8\delta}$$

However, g'(x) becomes complex for $\delta \geq \frac{1}{8}$