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CS 485

Random graph with a given degree distribution.

$$\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} (d_1 \dots d_n) = \begin{pmatrix} d_1 d_1 & d_1 d_2 & \dots & d_1 d_n \\ \vdots & & \ddots & \vdots \\ d_1 d_n & \dots & \dots & d_n d_n \end{pmatrix} \to 0/1$$

randomly round to

Depending on d_i, there may or may not be a giant component

Condition:

$$\sum_{i=1}^{n} i(i-2)\lambda_i = 0 \leftarrow \text{if } >0 \text{, there is a giant component.}$$

where λ_i is the frequency of a degree i vertex.

For $G(n,p)$

$$\lambda_i = \binom{n}{i} p^i (1-p)^{n-i}$$

$$\sum_{i=0}^{n} i(i-2) \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=0}^{n} i(i-2) \frac{(n(n-1)\dots(n-i))}{i!} \frac{1}{n^i} \left(1-\frac{1}{n}\right)^{n-i}$$

$$= \frac{1}{e} \sum_{i=0}^{n} i(i-2) \frac{(n)(n-1)\dots(n-i)}{i!\,n^i} \left(\frac{n-1}{n}\right)^{-i}$$

$$= \frac{1}{e} \sum_{i=0}^{n} \frac{i(i-2)}{i!} \frac{n(n-1)\dots(n-i)}{(n-1)(n-1)\dots(n-1)}$$

$$\le \frac{1}{e} \sum \frac{i(i-2)}{i!}$$

see that $\displaystyle\sum_{i=0}^{n} \frac{i}{i!} = \sum_{i=1}^{n} \frac{i}{i!} = \sum_{i=1}^{n} \frac{1}{(i-1)!} = \sum_{j=0}^{n-1} \frac{1}{j!}$

Now look at $\displaystyle\sum_{i=0}^{n} \frac{i^2}{i!} = \sum_{i=1}^{n} \frac{i}{(i-1)!} = \sum_{j=0}^{n-1} \frac{j+1}{j!} = \sum_{j=0}^{n-1} \frac{j}{j!} + \sum_{j=0}^{n-1} \frac{1}{j!}$

$$= \sum_{j=1}^{n-2} \frac{1}{j!} + \sum_{j=0}^{n-1} \frac{1}{j!}$$

Going back

$$\sum_{i=0}^{n} \frac{i(i-2)}{i!} = \sum_{j=0}^{n-2} \frac{1}{j!} + \sum_{j=0}^{n-1} \frac{1}{j!} - 2\sum_{j=0}^{n-1} \frac{1}{j!} = \frac{1}{(n-1)!} \to 0$$

<u>Exponential degree distribution</u>

1) Start with a single vertex

2) Each time you want to add a vertex + δ edges.

$$\lambda_i = \frac{(2\delta)^i}{(1+2\delta)^{i+1}} \qquad \delta = \frac{1}{4} \qquad \lambda_i = \frac{2}{3}\left(\frac{1}{3}\right)^i$$ which is indeed exponential

$$\sum_{i=0}^{\infty} i(i-2)\frac{2}{3}\left(\frac{1}{3}\right)^i = \frac{2}{3}\sum_{i=0}^{\infty} i^2\left(\frac{1}{3}\right)^i - 2\sum_{i=0}^{\infty} i\left(\frac{1}{3}\right)^i$$

$$= \frac{2}{3}\left[\frac{a(1+a)}{(1-a)^3} - \frac{2a}{(1-a)^2}\right] = \frac{2}{3}\left[\frac{1/3 \cdot 4/3}{(\frac{2}{3})^3} - \frac{2(\frac{1}{3})}{(\frac{2}{3})^2}\right]$$

$$= \frac{2}{3}\left[\frac{3}{2} - 2\left(\frac{3}{4}\right)\right] = 0$$

So if there is a random graph with given degree distribution, then there is a phase transition.

$$\#\text{of edges} = \frac{1}{2}\sum_{i=0}^{\infty}(\lambda_i n)i = \frac{n}{2}\sum_{i=0}^{\infty} \underbrace{\frac{2}{3}\left(\frac{1}{3}\right)^i}_{\lambda_i}$$

$$= \frac{n}{3}\sum_{i=0}^{n}\left(\frac{1}{3}\right)^i$$

$$= n/3\left[\frac{1/3}{(1-1/3)^2}\right] = \frac{n}{4}$$

Let $d_k(t)$ be the expected number of vertices of degree k at time t

$$d_0(t+1) = d_0(t) + 1 - 2\delta \frac{d_0(t)}{t}$$

$$d_k(t+1) = d_k(t) + 2\delta \frac{d_{k-1}(t)}{t} - 2\delta \frac{d_k(t)}{t}$$

Consider a solution of form $d_k(t) = p_k t$

$$(t+1) p_0 = p_0(t) - 2\delta p_0 + 1$$

$$p_0 = \frac{1}{1+2\delta}$$

$$(t+1) p_k = p_k t + 2\delta p_{k-1} - 2\delta p_k$$

$$p_k = \frac{2\delta p_{k-1}}{1+2\delta} = \left(\frac{2\delta}{1+2\delta}\right)^k \left(\frac{1}{1+2\delta}\right)$$