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There is a whole host of generating functions and we will be discussing one more type known as the exponential generating function.

Exponential Generating Function

$a_0, a_1, a_2 \leftrightarrow g(x) = \sum_{i=0}^{\infty} a_i \left(\frac{x^i}{i!}\right)$

Moment Generating Functions

$E(x^k)$ is the k^{th} moment about the origin

> *Clarification: These moments tell us about integrating the k^{th} power. For example, the first moment is the average. The second moment is the squared distance away from the origin. If you have all of the moments, then you have the pdf.*

$$\psi(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} p(x)dx = \int_{-\infty}^{\infty} \left[1 + tx + \frac{(tx)^2}{2!} + \ldots\right] p(x)dx$$

The k^{th} moment of x is k! times coefficient of t^k in the moment generating function.

> *Explanation: Fourier Transforms: transforms one domain to another. An example is representing music in terms of its sound frequency.*

Probability distribution p(x) $\leftrightarrow$ $\psi(t)$ *//Function of time to function of frequency*

Fourier Transform

$$\int_{-\infty}^{\infty} e^{tx\sqrt{-1}} p(x)dx$$

The moment generating function has all of the properties of the Fourier transform.

One usage of the Fourier transform with respect to distribution is the following:

Gaussian Probability Distribution

Assume mean = 0 and unit variance ($\sigma^2 = 1$),

$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

First, we calculate the moments:

$$\mu_n = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} dx$$

$$\mu_n = \begin{cases} \dfrac{n!}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!} & \text{n even } \textit{//Use by parts to get recurrence relation: } \mu_n = (n-1)\mu_{n-2} \\ 0 & \text{n odd } \textit{//Because } e^{-\left(\frac{x^2}{2}\right)} \textit{ is symmetric} \end{cases}$$

$\mu_0 = 1$

$\mu_1 = 0$

$$g(s) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} s^n$$

$$= \sum_{n=0\&even}^{\infty} \frac{n!}{2^{\frac{n}{2}}\left(\frac{n}{2}\right)!} \frac{1}{n!} s^n$$

To change the indices, let $n = 2i$,

$$= \sum_{i=0}^{\infty} \frac{s^{2i}}{2^i (i)!} = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{s^2}{2}\right)^i$$

since $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, g(x) = $e^{\frac{s^2}{2}}$ *//This is the moment generating function for Gaussian*

In general g(s) = $e^{s\mu + \frac{\sigma^2 s^2}{2}}$

Question: What is the probability distribution of the sum of two Gaussian probability distributions?

x_1 $\quad \mu_1, \sigma_1 \leftrightarrow e^{s\mu_1 + \frac{\sigma_1^2 s^2}{2}}$

x_2 $\quad \mu_2, \sigma_2 \leftrightarrow e^{s\mu_2 + \frac{\sigma_2^2 s^2}{2}}$

$x_1 + x_2$ $\quad e^{(\mu_1+\mu_2)s + \frac{(\sigma_1^2+\sigma_2^2)s^2}{2}} + e^{s\mu_2 + \frac{\sigma_2^2 s^2}{2}}$

Conclusion: Result is Gaussian even if you add the two.

We now need to know the Catalan of numbers via the generating functions.

<u>Catalan Numbers</u>

Balanced parentheses of length 2n:

$c_0 = 1$	ε
$c_1 = 1$	( )
$c_2 = 2$	( ) ( ) , (( ))
$c_3 = 5$	((( ))) , (( ) ( )) , ( )( )( ), (( ))( ), ( ) (( ))

The general structure is $\underbrace{(\ \underbrace{A}_{j}\ )\ \underbrace{B}_{i-j-1}}_{i \text{ pairs}}$

$c_i = c_0c_{i-1} + c_1c_{i-2} + \ldots + c_{i-1}c_0$

//Convolution of sequence suggests squaring

Now let c(x) = $\sum_{i=0}^{\infty} c_i x^i$

$$c^2(x) = \sum_{i=0}^{\infty} c_i x^i \sum_{j=0}^{\infty} c_j x^j = {c_0}^2 + (c_0c_1 + c_1c_0)x + (c_0c_2 + c_1c_1 + c_2c_0)x^2 + \ldots$$

Substituting c_i for $c_0c_{i-1} + c_1c_{i-2} + \ldots$ we get

$c^2(x) = c_1 + c_2x + c_3x^2 + \ldots$
$c_0 + xc^2(x) = c_0 + c_1x + c_2x^2 + c3x^3 + \ldots$
$c_0 + xc^2(x) = c(x)$

Substituting $c_0 = 1$ and solve for c(x) yields

$xc^2(x) - c(x) + 1 = 0$ *//Use quadratic formula*

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Minus sign gives correct answer, so c(x) = $\frac{1 - \sqrt{1-4x}}{2x}$

We used $(1-y)^{1/2}$ =

… *//Refer to the last page for detailed calculations*

$$c(x) = \sum_{i=0}^{\infty} \frac{1}{1+i}\binom{2i}{i}x^i$$

$$\boxed{c_i = \frac{1}{i+1}\binom{2i}{i}}$$

Catalan numbers are used in calculating the eigen value distributions, which is semi-circular (1920, Physicist Wigner).

Alternative Approach: Let us look at <u>number of strings of length 2n</u> with equal number of left and right parentheses is $\binom{2n}{2}$ //It is easiest to calculate the number of strings //that aren't balanced and subtract them off.

Each of these strings is balanced unless there is a prefix with one more right than left parentheses.

$\underbrace{(\,(\,)\,)\,)}\,\underbrace{(\,(\,)}_{\text{flip}}$

(( ))))) (

Flip left to right, right to left

n+1 right parentheses
n-1 left parentheses

$$c_n = \binom{2n}{n} - \binom{2n}{n+1}$$
$$= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} = \frac{(2n)!(n+1-(n))}{n!(n+1)!}$$
$$= \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}$$

$c_i = \frac{1}{1+i}\binom{2i}{i}$, which is the same result obtained from the other method.

<u>EXTRA: Details from the quadratic calculation</u>

Minus sign gives correct answer, so c(x) = $\frac{1-\sqrt{1-4x}}{2x}$

$$(1-4x)^{1/2} = 1 - \frac{1}{2}4x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)(4x)^2}{2!} - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(4x)^3}{3!} + \ldots$$

$$= 1 - 2x + \frac{2^2x^2}{2!} - \frac{3(2x)^3}{3!} - \ldots - \frac{3*5*\ldots*(2n-3)2^n x^n}{n!}$$

Since 1 * 3 * 5 * … * (2n-3) = $\frac{(2n)!}{(2n-1)2^n n!}$

$$= 1 - \sum_{n=1}^{\infty} \frac{(2n)!}{(2n-1)2^n n!} \frac{2^n x^n}{n!}$$

Thus c(x) $= \frac{1}{2x}\sum_{n=1}^{\infty} \frac{(2n)!x^n}{(2n-1)n!n!}$

$$= \sum_{n=1}^{\infty} \frac{(2n)(2n-1)(2n-2)!x^{n-1}}{2(2n-1)n^2(n-1)!(n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{(2n-2)!x^{n-1}}{n(n-1)!(n-1)!}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1}x^{n-1}$$

Substituting i = n-1, gives you

c(x) = $\sum_{i=0}^{\infty} \frac{1}{1+i}\binom{2i}{i}x^i$