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Generating Function

Branching process:

Let P_i be the probability of i children.

Let $g(x) = \sum P_i X^i$ be the corresponding generating function.

Define jth iteration of g(x)

$$g_1(x) = g(x)$$

$$g_2(x) = g(g(x))$$

.

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$$g_j(x) = g_{j-1}(g(x))$$

Two observations:

$g^2(x)$ is the generating function for the sum of two independent random variables x_1+x_2 where x_1 and x_2 have probability distribution Pi.

$$g^2(x) = P_0^2 + (P_0P_1 + P_1P_0)x + (P_1P_2 + P_1P_1 + P_2P_0)x^2 + \dots$$

For x_1+x_2 to have value 0 both x_1 and x_2 must have value zero.

For x_1+x_2 to have value 1 exactly one of x_1, x_2 must have value 1 and then other have value 0.

In generating $g^r(x)$ is the generating function for $x_1+x_2+...+x_r$

$g_j(x)$ is generating function for number of children in jth generation of branching process.

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Proof

By induction on j,

By Induction Hypothesis,

$$g_{j+1}(x) = b_0 + b_1x + b_2x^2 + \ldots + b_ix^i + \ldots$$

where coefficient of x^i is probability of i children in j-1 level.

If i children in j-1 generation, these will contribute in total $g^i(x)$

$$g_j(x) = b_0 + b_1g(x) + b_2g^2(x) + \ldots + b_ig^i(x)$$

$$= g_{j-1}(g(x))$$

Generating function for sequence defined by recurrence relationship.

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⇨ e.g. Fibonacci sequence.

$F_0 = 1, \quad F_1 = 1, \quad F_i = F_{i-1} + F_{i-2} \quad (i>=2)$

How do we get generating function for Fibonacci sequence?

$$f_i x^i = f_{i-1} x^i + f_{i-2} x^i \quad (\text{*xi on both sides})$$

$$\sum(_{i=2,\infty}) f_i x^i = \sum(_{i=2,\infty}) f_{i-1} x^i + \sum(_{i=2,\infty}) f_{i-2} x^i$$

$$\text{Let } f(x) = \sum(_{i=0,\infty}) f_i x^i$$

$$f(x) - f_1 x = x f(x) + x^2 f(x)$$

$$f(x) - x f(x) - x^2 f(x) = x \quad (\text{by rearranging})$$

$$f(x) = x / (1 - x - x^2)$$

Asymptotic Behavior

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$$f(x) = (\sqrt{5}/5) / (1 - Ø_1 x) \quad + (\sqrt{5}/5) / (1 - Ø_2 x)$$

where $Ø_1 = (1+\sqrt{5}) / 2$, $Ø_2 = (1-\sqrt{5}) / 2$

$$f(x) = (\sqrt{5}/5) [1 + Ø_1 x + (Ø_1 x)^2 + \ldots -(1 + Ø_2 x + (Ø_2 x)^2 + \ldots)]$$

$$f_n = (\sqrt{5}/5) (Ø_1 n - Ø_2 n) \qquad | Ø_2 | < 1$$

$$f_n = \lfloor (\sqrt{5}/5) \rfloor \; Ø_1^n$$

$$\lfloor f_n + (\sqrt{5}/5) Ø_2^n \rfloor = (\sqrt{5}/5) Ø_1^n$$

Where $\lfloor \; \rfloor$ sign is round down sign.

<u>Mean</u>

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Let z be an integer valued random variable. Let p_i be probability that z=i

$$E(z) = \sum_{(i=0,\infty)} i p_i$$

$$\text{Let } p(x) = \sum_{(i=0,\infty)} p_i x^i$$

$$p'(x) = \sum_{(i=0,\infty)} i p_i x^{i-1}$$

$$x p'(x) = \sum_{(i=0,\infty)} i p_i x^i$$

$$p'(1) = \sum_{(i=0,\infty)} i p_i \quad \leftarrow \text{mean}$$

Exponential Generating function

$$a0a1a2\ldots \quad \leftrightarrow g(x) = \sum_{(i=0,\infty)} a_i \, (x^i/i!)$$