Lecture 11: Generating Functions

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1 Generating Functions

Suppose we are given a sequence of numbers $a_0, a_1, a_2, ...$ We associate a single entity, a *generating function* $g(x)$, to represent this infinite sequence:

$$a_0,\ a_1,\ a_2, ... \Longleftrightarrow g(x) = \sum_{i=1}^{n} a_i x^i$$

For instance, the generating function for $a_0 = a_1 = ... = 1$ is

$$1,\ 1,\ 1,\ ... \Longleftrightarrow \sum_{i=1}^{n} x^i = \frac{1}{1-x}$$

Suppose we differentiate the generating function:

$$\frac{d}{dx}\ g(x) = \sum_{i=0}^{\infty} i\ a_i\ x^{i-1}$$

Multiply both sides by x:

$$\frac{d}{dx}\ g(x) = \sum_{i=0}^{\infty} i\ a_i\ x^i \iff 0,\ a_1,\ 2a_2,\ 3a_3,\ 4a_4,\ ...$$

Let $g(x) = (1-x)^{-1}$ as above. Then since $\frac{d}{dx}(1-x)^{-1} = (1-x)^{-2}$ we know

$$x\frac{d}{dx}(1-x)^{-1} = x(1-x)^{-2} = \frac{x}{(1-x)^2} \Longleftrightarrow \text{Generating function for } 0, 1, 2, 3, 4, ...$$

2 Powers

Let $g(x) = \sum_{i=0}^{\infty} g_i x^i$ where g_i are the probabilities that an integer valued random variable has value i.

Suppose we have two independent random variables $x_1,\ x_2$ that we want to add together such that

$$Prob(x_1 + x_2) = i$$

$$\Longrightarrow Prob(x_1 = 0)\ Prob(x_2 = i)\ +\ Prob(x_1 = 1)\ Prob(x_2 = i - 1) + ...$$

$$= g_0 g_i + g_1 g_{i-1} + g_2 g_{i-2} + ... + g_i g_0$$

You might recognize the above sum as something called a *convolution.*

Now we try to find $g^2(x)$:

$$g^2(x) = (\sum_{i=0}^{\infty}\ g_i\ x^i)(\sum_{j=0}^{\infty}\ g_j\ x^j)$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} g_i\ g_j\ x^{i+j}$$

Let $k = i + j$. Then

$$= \sum_{k=0}^{\infty}[\sum_{j=0}^{k}\ g_{k-j}\ g_j\ ]x^k$$

Note in brackets the coefficients of x^k are grouped as powers of x.

$$\Rightarrow \textstyle\sum_{j=0}^{k} g_{k-j} g_j \Longleftrightarrow \underbrace{g_0 g_0}_{k=0},\ \underbrace{g_1 g_0 + g_0 g_1}_{k=1},\ \underbrace{g_2 g_0 + g_1 g_1 + g_0 g_2}_{k=2},\ ...$$

3 Examples

Example 1:

Suppose we are given three object types A, B and C:

A	can be selected 0 or 1 times
B	can be selected 0, 1, or 2 times
C	can be selected 0, 1, 2, or 3 times

How many ways can you select five objects?

We calculate the generating functions for A, B and C as:

A	$1 + x$
B	$1 + x + x^2$

C $\quad 1 + x + x^2 + x^3$

To find the total number of ways to select A, B and C, we multiply their respective generating functions together:

$$(1+x)(1+x+x^2)(1+x+x^2+x^3)$$
$$= 1 + 2x + \underbrace{5x^2}_{check} + 6x^3 + 5x^4 + \underbrace{3x^5}_{check} + x^6$$

We check the term $5x^2$. There are five ways to pick two objects as follows:

$$CC,\ BB,\ CB,\ CA,\ BA$$

We also check the term $3x^5$. There are three ways to pick five objects as follows:

$$CCCBB,\ CCCBA,\ CCBBA$$

Example 2:

Suppose we try to make change with pennies, nickels and dimes.

Pennies	$1 + x + x^2 + x^3 + x^4 + ...$
Nickel	$1 + x^5 + x^{10} + x^{15} + ...$
Dimes	$1 + x^{10} + x^{20} + ...$

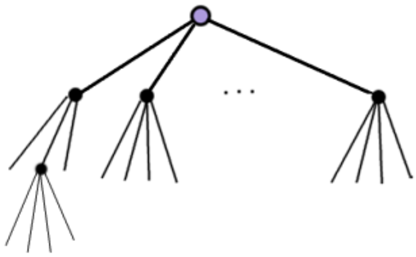
To find the change possibilities, we multiply the coins' generating functions together:

$$(1+x+x^2+...)(1+x^5+x^{10}+...)(1+x^{10}+x^{20}+...)$$
$$= 1 + x^2 + x^3 + x^3 + x^4 + 2x^5 + ... + \underbrace{9x^{23}}_{check} + ...$$

We want to check the term $9x^{23}$. There are nine ways to make change for 23 cents as follows:

Pennies	3	3	8	13	3	8	13	18	23
Nickel	0	2	1	0	4	3	2	1	0
Dimes	2	1	1	1	0	0	0	0	0

4 Branching Trees



Why are branching trees important?

Hereditary - Given a family tree, will that family's bloodline die out?

Epidemics - We can model how a disease spreads using a branching tree. Will a given epidemic spread throughout the world or die out (relatively) harmlessly?

Search in a random graph - When performing a breadth-first search from a node n, what are the expected number of children reachable from n? Will that search eventually die out? What is the expected graph component size?

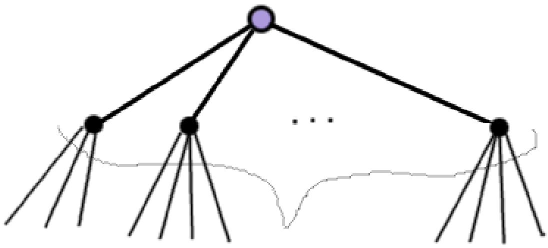
Let $p(x)$ be the probability that x takes on a given value. Let $f(x)$ be the generating function for probability of i children.

Then in the j^{th} generation, the number of children is found to be:

$$f_j(x) = \begin{cases} f(x) & j = 1 \\ f_{j-1}(f(x)) & j > 1 \end{cases}$$

Show by induction:

Coefficient of x^r in $f_{j-1}(x)$ is the probability of r children in $(j-1)^{\text{st}}$ generation.



Each of these r children contribute $f(x)$ to j^{th} generation.

For a sum of r sets of descendants, the generating function is $f^r(x)$. Why? Any sum of r random variables has a generating function $f^r(x)$ because if $x \Longleftrightarrow g(x)$ then $\underbrace{x + ... \ + x}_{r} \Longleftrightarrow g^r(x)$ by convolution argument.

We know the total contribution of r children is $f^r(x)$.

Substitute in $f_{j-1} = a_0 + a_1 x \ + a_2 x \ + ...$

Substitute in $f^r(x)$ for x^r:

$$= f_{j-1}(f(x))$$

Assuming coefficient of x^r in f_{j-1} give correct probability for r children in $(j-1)^{st}$ generation, then substituting $g(x)$ for x gives generation for j^{th} generation.