Lecture Notes for CS 485, 2/13/06

Proof that every increasing property in the N_p system has a threshold.

Let Q be any increasing property and $P(\delta)$ be the value which makes $\text{Prob}(N_{P(\delta)} \in Q) = \delta$. Now consider m samples drawn from $N_{P(\delta)}$ and construct their union such that if any integer appears in any of the m samples drawn from $N_{P(\delta)}$, then it also appears in the union. For example:

$$[1, 10, 12]_1 \cup [3, 5, 7]_2 \cup \ldots \cup [1, 5, 6]_m = [1 \ldots 3 \ldots 5, 6, 7 \ldots 10 \ldots 12 \ldots]$$

Notice that the new set behaves as if drawn directly from an ensemble of type N_p, but with a higher value for p than $P(\delta)$. Call the union N_q. The probability of N_q having any given integer is $q = 1 - \text{Prob}(\text{none of the samples contain the integer})$
$= 1 - (1 - P(\delta))^m$

Since Q is increasing, if any sample has property Q, then N_q must also have Q. Also, N_q may have Q, even if none of the samples do. Therefore

$$\begin{aligned}\text{Prob}(N_q \notin Q) &\le \text{Prob}(\text{none of the } m \text{ samples from } N_{P(\delta)} \text{ has } Q) \\ &\le (1 - \text{Prob}(N_{P(\delta)} \in Q))^m \\ &\le (1-\delta)^m \quad \text{since} \quad \text{Prob}(N_{P(\delta)} \in Q) = \delta \text{ by def.}\end{aligned}$$

Now choose δ to be any number between 0 and $\frac{1}{2}$, and m to be any number large enough that $(1-\delta)^m \le \delta$. Then

$\text{Prob}(N_q \notin Q) \le (1-\delta)^m \le \delta$ so that

Eq 1: $\text{Prob}(N_q \in Q) = 1 - \text{Prob}(N_q \notin Q) \ge 1 - \delta = \text{Prob}(N_{P(1-\delta)} \in Q)$

Note that $q = 1 - (1 - P(\delta))^m \le m P(\delta)$ for $m \ge 1$.

Because Q is an increasing property, then

$\text{Prob}(N_q \in Q) \le \text{Prob}(N_{mP(\delta)} \in Q)$

After combing this information with Eq 1, we get

$\text{Prob}(N_{P(1-\delta)} \in Q) \le \text{Prob}(N_{mP(\delta)} \in Q)$ so that

$P(1-\delta) \le m P(\delta)$.

Because $\delta < \frac{1}{2}$ then $\delta < 1 - \delta$, so that

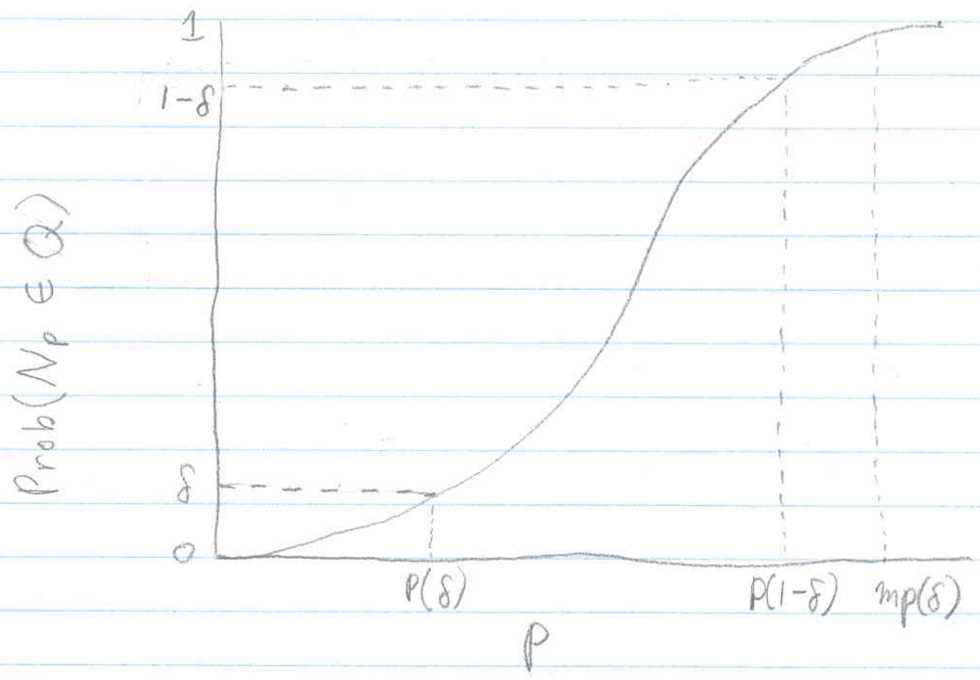
$P(\delta) \le P(1-\delta) < m P(\delta)$

which proves that Q has a threshold.

A property Q has a threshold on the variable p (which is a function of the system size, n) if for any δ, $0<\delta<\frac{1}{2}$, the value of p required to make $Prob(N_p \in Q) = \delta$ may be multiplied by a constant, m, to make $Prob(N_{mp} \in Q) \geq 1-\delta$. Therefore, no matter how close δ gets to 0 or $1-\delta$ gets to 1, the transition of the system from a low probability of having Q, δ, to a high probability of having Q, $1-\delta$, happens within a constant factor, m. This can be stated as

$$\forall\ 0>\delta>\tfrac{1}{2},\ \exists\, m \text{ s.t. } P(\delta) \leq P(1-\delta) \leq mP(\delta)$$

This can be seen graphically as



Monday February 13 Notes Part 2

Back to cliques:

Consider G(n, $\frac{1}{2}$). We showed earlier that you can clearly find a clique of size log n. There is also a clique of size 2 log n but we don't have an algorithm to find it.

Matula: took 165 graphs, each with 32 vertices and Prob(edge existing between vertex i and vertex j)= $\frac{1}{2}$. This is what he found:

Clique size	5	6	7	8
# of graphs	1	90	68	8

As you can see, the size of the cliques are very highly concentrated.

Let us rephrase the first sentence above:

For any ε > 0 almost surely G(n, ½) has a clique of size (2-ε) log n.
There is almost surely no clique of size 2 log n.

Proof:

let f(k) be the expected # of cliques of size k.

Let f(k) be the expected number of cliques of size k.

$$f(k) = \binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}.$$

Now we need to prove

$$\lim_{n \to \infty} \text{f(2 log n)} = 0$$

and

$$\lim_{n \to \infty} \text{f[(2-ε)]} = \infty \text{ (by second moment)}$$

f(2 log n)= $$\binom{n}{2\log n}\left(\frac{1}{2}\right)^{2\log^2 n}$$

$$= \frac{n^{2\log n}}{(2\log n)!}\frac{1}{(2^{2\log n})^{2\log n}}$$

$$= \frac{n^{2\log n}}{(2\log n)!*n^{2\log n}}$$

$$= \frac{1}{(2\log n)!} \to 0$$

$$\text{f[(2-ε) log n]} = \binom{n}{(2-\varepsilon)\log \text{n}} \left(\frac{1}{2}\right)^{\frac{(2-\varepsilon)^2 \log^2 n}{2}}$$

$$= \frac{n^{(2-\varepsilon)\log n}}{((2-\varepsilon)\log n)!} \frac{1}{n^{((2-\varepsilon)^2/2)\log n}}$$

$$= \frac{n^{\frac{\varepsilon}{2}(2-\varepsilon)\log n}}{((2-\varepsilon)\log n)!}$$

- ⇨ notice that the top grows faster than the bottom (try taking the log of each).
- ⇨ Therefore this limit is infinity.