NP-Complete Graph Problems

Announcement: PSet 7 to be released Friday morning.

Q1 will have optional "early hand-in."

due Tues 4pm

promise to grade before end of Sp Brk,

solution set for entire PSet 7

will be released to all

couple of days after latest slip-day deadline)

regardless of early Q1 option.

Rest of PSet 7 has usual Thurs night

deadline.

Recall.

3SAT. Given boolean variables \( x_1, \ldots, x_n \)

forming literals \( x_i, \overline{x_i}, x_i \lor \overline{x_i}, \ldots, x_n, \overline{x_n} \)

and clauses \( C_1, \ldots, C_m \)

each is disjunction (Boolean OR)

of \( \leq 3 \) literals.

... does a truth assignment satisfying

all clauses exist?

IND SET. Given undirected graph \( G \)

pos. integer \( k \)

... does exist set \( S \) of \( k \) vertices,

st. every edge has \( \leq 1 \) endpoint

in \( S \)?

Claim. 3SAT \( \leq_p \) IND SET

"Given an algorithm that solves IND SET
we can make an efficient 3SAT algorithm.

Goal. ① Transform input of 3SAT \( \equiv \) input of IND SET.

② Transformation runs in poly time.
   (In fact, will be linear time.)

③ If 3SAT input has a satisfying truth assignment, the IND SET instance will have a k-element ind set.

④ If 3SAT input has no satisfying assign.
   the IND SET instance will have no k-element independent set.

\[
C_1: x_1 \lor x_2 \lor \overline{x_3} \\
C_2: \overline{x_1} \lor \overline{x_2} \lor x_4
\]
In general, the reduction takes

\[ \begin{align*}
\text{Variables} & \quad \rightarrow \quad 2n \text{ verts} \\
\quad x_1, \ldots, x_n & \\
\text{Clauses} & \quad \rightarrow \quad \leq 3m \text{ verts} \\
\quad C_1, \ldots, C_m & \quad \rightarrow \quad \leq 3m \text{ verts}
\end{align*} \]

\[ \begin{align*}
\text{edge set:} & \quad \text{Connect } (u_j, v_j) \quad \forall j \in [n] \\
\quad (w_{ij}, w_{ik}) & \quad \forall j \neq k \text{ s.t. } \{x_j, \overline{x_j}, x_k, \overline{x_k}\} \text{present in } C_i \\
\quad (w_{ij}, u_j) & \quad \text{if } x_j \text{ is in } C_i \\
\quad (w_{ij}, v_j) & \quad \text{if } \overline{x_j} \text{ is in } C_i
\end{align*} \]

Set target indep set size \( k \), to be \( n+m \).

**Running time:** \( O(2^n + 3m + n + 6m) = O(m+n) \)

**CLIQUE:** Given (undir.) graph \( G \) and \( k \in \mathbb{N} \), does \( G \) have a set of \( k \) vertices, \( S \), such that every two elements of \( S \) are connected by an edge?
**Vertex Cover:** Given $G$ and $k$, does $G$ have a set of vertices $S$, with $k$ elements, that "covers" every edge? ($S$ contains at least one endpoint of every edge)

**IND-SET $\leq_p$ CLIQUE**

Given $G = (V,E)$ and $k$,

construct $\overline{G} = (V, \overline{E})$ and $k$.

$S$ is independent in $G \iff$

$S$ is clique in $\overline{G}$. 