Announcement. Final exam scheduling poll
Al time slots on May 20–24.
Link to poll is in a pinned post on Ed.
Responses due Saturday (tomorrow) night, 23:59.
No response $\Rightarrow$ May 25, 9:30 a.m.

Today is last day to drop courses

Global Min Cut in Undirected Graphs

Undirected $G = (V,E)$.
All edges have capacity 1.
Goal. Find a partition $V = A \cup B$ with
$A$ and $B$ non-empty such that
$c(A,B)$ is minimized.

Distinct from the minimum cut
problem where we require $s \in A, t \in B$. 
Example application: You want to make sure that your fuel pipeline network would remain connected even if a cyberattacker shut down $k$ pipelines...
compute global min cut, test if $|A \cup B| < k$.

One way to solve: reduce to many instances of $s^t$ min cut.

Ex. for all $s$ in $V$:
   for all $t$ in $V \backslash \{s\}$:
      compute $s^t$ min cut
      output best cut discovered,

This takes $n \cdot (n-1) \cdot T(\text{min cut})$.
If $m = \# \text{ edges}$, $n = \# \text{ vertices}$,
Ford-Fulkerson compute one $s^t$ min cut
in $O(mn)$ $\implies$ running time $O(m^3n^3)$.

Faster: Choose any $s$ in $V$.
   For all $t$ in $V \backslash \{s\}$:
      compute $s^t$ min cut
      output best cut discovered,

Improve running time to $O(m^3n^2)$. 
Karger's Randomized Contract Algorithm

A multigraph is a graph that may have 2 or more edges between the same pair of vertices.

If $G$ is a multigraph and $e$ is an edge of $G$, $e = (u,v)$, then

$\text{CONTRACT}(G,e)$ is an operation that outputs new graph $G'$ with

$$V(G') = (V(G) \setminus \{u,v\}) \cup \{w\}$$

$$E(G') = \left\{ \begin{array}{l}
(x,y) \in E(G) : x,y \notin \{u,v\} \\
(x,w) \big| (x,u) \in E(G), x \notin \{u,v\} \\
(x,w) \big| (x,v) \in E(G), x \notin \{u,v\}
\end{array} \right\}$$

In words: merge $u,v$ together into a new node $w$. The neighbors of $w$ are all the neighbors of $uv$ combined.

If $G'$ has edges between $u$ & $v$ delete them.

$$G \xrightarrow{\text{CONTRACT}(G,e)} G'$$
Karger's Algorithm

repeat S times: // S is a parameter that determines success probability

let \( H \) = fresh copy of \( G \)

while \( H \) has more than 2 vertices:
  sample random edge \( e \in E(H) \)
  replace \( H \) with \( \text{CONTRACT}(H, e) \)
  // now \( H \) has 2 vertices.

let \( a, b \) be the 2 vertices of \( H \).

let \( A = \{ \text{vertices of } G \text{ contracted to } a \} \)

let \( B = \{ \text{vertices of } G \text{ contracted to } b \} \)

store \((A, B)\) in our collection of potential min cuts.

output the cut with smallest capacity among potential min-cuts discovered.

Implementation: \( \text{CONTRACT}(H, e) \) can be done in \( O(\log n) \) time per operation using \( \text{union-find} \) to implicitly store \( V(H) \).

Find each endpoint of \( e \), take union of these 2 sets.
One iteration of main loop: $O(n \log n)$
All $s$ iterations: $O(n s \log n)$

Success probability? The hope is that we make it through $n-1$ iterations of edge contraction without ever contracting an edge of the min-cut, $(A^*, B^*)$.

Let $k = c(A^*, B^*)$.
Consider an iteration where we have a contracted graph $H$ with $m_H$ edges, $n_H$ vertices, and suppose that no edge from $A^*$ to $B^*$ is yet contracted.

$\implies$ min-cut capacity of $H = k$. 
Degree of every vertex in $H \geq k$.

$$2 \cdot m_H = \sum \text{degrees of vertices in } H \geq n_H \cdot k$$

$$\frac{k}{m_H} \leq \frac{2}{n_H}.$$ 

Prob of randomly sampling an edge from $A^*$ to $B^*$.

$$\text{Prob (an edge from } A^* \text{ to } B^* \text{ is not sampled)}$$

$$\geq \frac{n_H - 2}{n_H}.$$ 

What is the probability of getting from $n$ vertices down to 2 without ever contracting an edge from $A^*$ to $B^*$?

It's at least

$$\left( \frac{n^2}{n} \right) \cdot \left( \frac{n-3}{n-1} \right) \cdot \left( \frac{n-4}{n-2} \right) \cdot \cdots \cdot \left( \frac{1}{3} \right)$$

$$= \frac{2}{n (n-1)}.$$ 

Each outer loop iteration succeeds
in finding $(\alpha^*, \beta^*)$ with prob
at least \( \frac{1}{(\frac{1}{2})^n} \).

Repeat outer loop \( s = \binom{n}{2} \cdot \ln(n) \) times.

\[ \text{Pr (none of the iterations succeed)} \leq \left( 1 - \frac{1}{(\frac{1}{2})^n} \right)^{\binom{n}{2} \cdot \ln(n)} \leq e^{-\binom{n}{2} \cdot \ln(n)} \leq e^{-\ln(n)} = \frac{1}{n}. \]

Running time \( O(s \cdot n \cdot \log n) = O(n^3 \log^2 n) \).

Karger & Stein found a beautiful divide-and-conquer way to improve the running time by factor \( n \).