Given an undirected graph $G = (V, E)$. A perfect matching $M \subset E$ is a subset of edges such that there is exactly one edge in M adjacent to any node $v \in V$. Note that G doesn't have to be bipartite. The PERFECT MATCHING problem is to decide if the input graph G has a perfect matching. We have seen an algorithm solving this problem in bipartite graphs (using flows). A more complex, but still polynomial time, algorithm can solve this problem in general graphs also. Here we show that a SAT solver can solve this, as a first demonstration of the power of SAT.

Theorem. PERFECT MATCHING $\leq$ SAT

Proof. Consider a graph G with n nodes V and m edges E. We will have a variable x_e associated with each edge e, with the ideas that $x_e = true$ will indicate that e is in the matching. This is m variables.

The clauses will be used to make sure our selected set of edges is a perfect matching. We add a clause for every vertex v that guarantees that at least one edge adjacent to v is picked:

$$\vee_{e \text{ adj } v}\, x_e \text{ for each node } v \in V \tag{1}$$

Next we have to make sure only one edge is picked adjacent to every node. To do this, consider any pair of edges e and f that share a node. We add for al such pairs the clause

$$\bar{x}_e \vee \bar{x}_f \text{ for each pair of adjacent edes } e \text{ and } f \tag{2}$$

Now let Φ denote the formula consisting of the clauses (1) and (2) above. Note that (1) has n clauses, one for every node $v \in V$, while (2) has at most $O(nm)$, as each of the m edges can share a node with at most $2(n-1)$ other edges, $n-1$ at most at either end of the edge. So the formula Φ has at most $O(mn)$ clauses, which is polynomial in n and m.

Claim. The formula Φ is satisfiable if and only if G has a perfect matching.

Proof. Suppose G has a perfect matching M. We can set $x_e = true$ for all edges $e \in M$ and false for all other edges. This truth setting will satisfy (1) as M has an edge adjacent to every node, and it satisfies (2) as it has only one edge adjacent to any node.

Similarly, if a truth assignment satisfies Φ, it most have at least one edge adjacent to every node due to (1), and cannot have two edges adjacent to any node, due to (2).

Next we show that SAT can also be used to solve the INDEPENDENT SET problem. Recall that the INDEPENDENT SET problem is given by an undirected graph and an integer k, and asks if G has an independent set of size k.

Theorem. INDEPENDENT SET $\leq$ SAT

Proof. Consider a graph G with n nodes V and m edges E. We will have a variable x_v associated with each node v, with the idea that $x_v = true$ will indicate that v is in the independent set. This is n variables. We can add a clause for each edge $e = (v, w)$

$$\bar{x}_v \vee \bar{x}_w \text{ for each edge } (v, w) \in E \tag{3}$$

which guarantees at most one of the two adjacent nodes is in the independent set. We summarize this as the following claim.

Claim. A truth assignment satisfies the clauses (3) if and only if the set $I = \{v : x_v = true\}$ is independent.

Next we need to make sure to that the independent set is size k. We can think of this as a matching problem, matching the nodes V picked by the independent set to a set of k nodes $\{1, \ldots, k\}$, as suggested by the figure below.

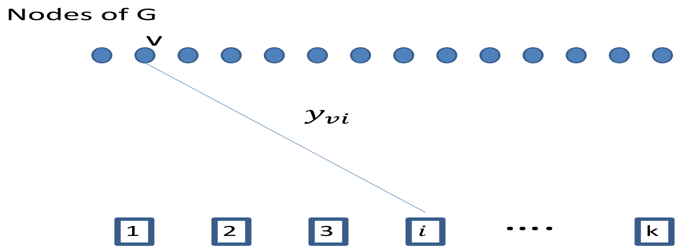

Figure 1: Matching Independent Set to nodes $1, 2, \ldots, k$.

Add new variables y_{vi} with the idea that $y_{vi} = true$ if v is the ith member of the independent set. Clauses will be analogous to the matching problem above. The first set guaranteeing that we pick an ith node for each $i = 1, 2, \ldots, k$.

$$\vee_v \, y_{vi} \text{ for each node } 1 \leq i \leq k \tag{4}$$

The next set of clauses makes sure each node v is only counted at most once. For each $i \neq j$ and each $v \in V$ we add

$$\bar{y}_{vi} \vee \bar{y}_{vj} \text{ for each node } v \text{ and } i \neq j \tag{5}$$

Finally, we need to make sure that only nodes in the independent set are picked. For each node v and each i we add

$$x_v \vee \bar{y}_{vi} \text{ for each node } v \text{ and each } i \tag{6}$$

Claim. There is a truth assignment satisfying all the above clauses if and only if the graph G has an independent set of size k.

Proof. To prove one direction, let I be an independent set of size k. We set $x_v = true$ for all nodes $v \in I$ and $false$ for all other nodes. Now number the nodes in I, and set $y_{vi} = true$ for the ith node, and all other y variables to $false$. We claim that this truth assignment satisfies all clauses above.

To see the opposite, consider a truth assignment satisfying all clauses. Let $I = \{v : x_v = true\}$. By the claim above I is a independent set. We need to claim that $|I| \geq k$. Note that it can actually be larger than size k, as we didn't include clauses guaranteeing that at most one edge (v, i) is selected for each node v.

For each $i = 1, \ldots, k$ there must be at least one variable y_{vi} that is true by (4). This node v must be in the independent set by (6), and each i must pick a different node v (or set of nodes) in the independent set due to (5), so the independent set is of size at least k.