# Reduction from 3 SAT to Subset Sum

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**Problem SUBSET SUM.** Given a set of integers $w_1, \ldots, w_n$ and a target sum $W$. The problem asks to decide if these is a subset $S \subset \{1, \ldots, n\}$ such that $\sum_{i \in S} w_i = W$. Please also read the discussion in Section 8.8 in the book about the role of large numbers in computation.

**Theorem.** SUBSET SUM is NP-complete.

First we note that SUBSET SUM is in NP. Given a set $S$, it takes up to $n$ additions to check that the sum $\sum_{i \in S} w_i$ is indeed equal to $W$, and addition can be done in polynomial time. The total time is $\sum_i O(\log w_i)$.

To prove that SUBSET SUM is NP-complete we will show that it is at least as hard as 3-SAT.

**Claim.** 3-SAT $\leq_P$ SUBSET SUM.

*Proof.* Consider a 3-SAT formula with $n$ variables $x_1, \ldots, x_n$ and $m$ clauses $c_1, \ldots, c_m$. We need to define numbers $w_i$ and a target sum $W$ that is equivalent to this 3-SAT problem. We will start by having two numbers $a_i$ and $b_i$ associated with each variable $x_i$ where including $a_i$ will correspond to setting $x_i$ true, and including $b_i$ will correspond to setting $x_i$ false. To do this, let

$$a_i = 10^{m+i} + \sum_{j: c_j \text{contains } x_i} 10^j$$

$$b_i = 10^{m+i} + \sum_{j: c_j \text{contains } \bar{x}_i} 10^j$$

Now consider a satisfying assignment, and the corresponding subset of the numbers so far, containing $a_i$ when $x_i = 1$ and containing $b_i$ when $x_i = 0$.

**Claim.** The resulting sum has the following form

- has a 1 in the leaning $n$ digits, corresponding to $10^{m+i}$ for $i = 1, \ldots, n$ as we included one of $a_i$ or $b_i$ for each $i$.
- 1, 2, or 3 in the next $m$ digits, the digit of $10^j$ is exactly the number of true literals in clause $j$, and that is nonzero if the assignment satisfies the 3-SAT formula.
- 0 in the final digit.

To turn this into a SUBSET SUM problem, we add a few more numbers. Let $W = \sum_{i=1}^{n} 10^{m+i} + 3 \sum_{j=1}^{m} 10^j$, and add $c_j = d_j = 10^j$ for $j = 1, \ldots, m$ to the numbers $a_i$ and $b_i$ defined above. We claim that this SUBSET SUM is solvable if and only if the 3-SAT is satisfiable.

- If 3-SAT satisfiable, select the number $a_i$ if $x_i = 1$ in the satisfying assignment, and $b_i$ if $x_i = 0$ in the satisfying assignment. By the claim above this gets the leading $n$ digits of $W$ correct. To make the remaining digits we may need to add $c_j$ or both $c_j$ and $d_j$ depending if the digit of $10^j$ is 3, 2, or 1.

- Finally, we need to prove that any solution to the subset sum problem corresponds to a solution to the 3-SAT problem. First note that for any digit $10^k$ there are at most 5 numbers with a 1 in that digits, the three literals corresponding to the clause $k$ (if $k \leq m$), and $c_k$ and $d_k$ (and at most 2 if $k > m$). With only 5 ones in any positions, so subset $S$ of these numbers will cause any carries in addition, so we can only get the total of $W$ by including the right number of 1s in any digit. To do this one needs to include exactly one of $a_i$ or $b_i$, and the corresponding truth assignment needs to satisfy the formula, so each position $1 \leq j \leq m$ we get at least one digit $10^j$. We can then add $c_j$ and $d_j$ to increase the digit to 3, as required by $W$.