

## Homework 7

**Reading:** Chapter 5.

Note that the due date is Tuesday at 3:00pm, not 8:30am. This is in order that you can make use of Tuesday office hours for the homework.

**Problem 1.** Suppose you're consulting for a bank that's concerned about fraud detection, and they come to you with the following problem. They have a collection of  $n$  bank-cards that they've confiscated, suspecting them of being used in fraud. Each bank-card is a small plastic object, containing a magnetic stripe with some encrypted data, and it corresponds to a unique account in the bank. Each account can have many bank-cards corresponding to it, and we'll say that two bank-cards are equivalent if they correspond to the same account.

It's very difficult to read the account number off a bank-card directly, but the bank has a high-tech "equivalence tester" that takes two bank-cards and, after performing some computations, determines whether they are equivalent.

Their question is the following: among the collection of  $n$  cards, is there a set of more than  $n/2$  of them that are all equivalent to one another? Assume that the only feasible operations you can do with the cards are to pick two of them and plug them in to the equivalence tester. Show how to decide the answer to their question with only  $O(n \log n)$  invocations of the equivalence tester.

You can assume for this problem that  $n$  is a power of 2.

**Problem 2.** In this problem, we will solve two recurrence relations.

(a) For  $T(n)$  given by the following recurrence, prove that  $T(n) = O(n \log n)$ . For this part, you don't have to prove that  $T(n) = \Omega(n \log n)$ .

$$T(n) = cn + \frac{1}{n} \sum_{i=0}^{n-1} (T(i) + T(n-1-i)),$$

$$T(1), T(2) \leq c,$$

$$T(0) = 0,$$

where  $c > 0$  is a constant.

**Hint:** Try substitution method, with  $T(n) = O(n \log n)$ . Note that you will need to specify exact constants, and you might need to add lower order terms. You might find the following identity useful:

$$\sum_{i=1}^{n-1} i \log i \leq \frac{n^2}{2} \log n - \frac{n^2}{2} + 2, \quad (0.1)$$

and you can use this identity without proof.

**Bonus problem:** Prove the above identity in (0.1).

(b) Solve for  $S(n)$ , where

$$S(n) = 2S(n/2) + cn \log n,$$

$$S(2) \leq c,$$

where  $c > 0$  is a constant. For this part, you must express your answer in  $\Theta(\cdot)$  notation. That is, if you happen to prove that  $T(n)$  should be of the order of  $n^d$ , then you must prove that it is  $O(n^d)$  as well as  $\Omega(n^d)$ .

**Problem 3.** Given a polynomial  $A(x) = \sum_{i=0}^{n-1} a_i x^i$  of degree bound  $n$ , its  $t$ -th derivative is defined by

$$A^{(t)}(x) = \begin{cases} A(x) & \text{if } t = 0, \\ \frac{d}{dx} A^{(t-1)}(x) & \text{if } 1 \leq t \leq n-1, \\ 0 & \text{if } t \geq n. \end{cases}$$

From the coefficient representation  $(a_0, a_1, a_2, \dots, a_{n-1})$  of  $A(x)$  and given a point  $x_0$ , we wish to determine  $A^{(t)}(x_0)$  for  $t = 0, 1, 2, \dots, n-1$ . We will solve this partially in this problem. (The solution is finished in the bonus problem below.)

(a) Given coefficients  $b_0, b_1, \dots, b_{n-1}$  such that

$$A(x) = \sum_{j=0}^{n-1} b_j (x - x_0)^j,$$

show how to compute  $A^{(t)}(x_0)$  for  $t = 0, 1, \dots, n-1$  in  $O(n)$  time. All  $n$  computations together should take  $O(n)$  time.

(b) Let us assume for the sake of this part that we know how to find  $A(x_0 + \omega_n^k)$  for  $k = 0, 1, \dots, n-1$ , where  $\{\omega_n^k : k = 0, 1, 2, \dots, n-1\}$  are  $n$ -th roots of unity. Explain how to find  $b_0, b_1, \dots, b_{n-1}$  in  $O(n \log n)$  time, given the values  $\{A(x_0 + \omega_n^k) : k = 0, 1, 2, \dots, n-1\}$ .

**Bonus problem:** In this problem, we will see how to actually compute  $\{A(x_0 + \omega_n^k) : k = 0, 1, \dots, n-1\}$ . Use the notation from the last problem.

(c) First prove that

$$A(x_0 + \omega_n^k) = \sum_{r=0}^{n-1} \left( \frac{\omega_n^{kr}}{r!} \sum_{j=0}^{n-1} f(j) g(r-j) \right),$$

where

$$f(j) = a_j \cdot j! \quad \text{and,} \\ g(l) = \begin{cases} \frac{x_0^{-l}}{(-l)!} & \text{if } -(n-1) \leq l \leq 0, \\ 0 & \text{if } 1 \leq l \leq (n-1). \end{cases}$$

(d) Explain how to evaluate  $A(x_0 + \omega_n^k)$  for  $k = 0, 1, \dots, n-1$  in  $O(n \log n)$  time. Conclude that all nontrivial derivatives of  $A(x)$  can be evaluated at  $x_0$  in  $O(n \log n)$  time.