Please remember to turn in each problem on a separate page, put your name on each page, and turn in the pages in three separate piles!

1. Give a NPDAs that recognize the following languages:

(a) The set of all strings in \{0, 1\}^* that contain twice as many 1s as 0s.

(b) The set

\[ \{ x\$$y \mid (\exists n) (x = \text{binary}(n) \land y = \text{binary}(n + 1)) \} \]

where \text{binary}(n) is the binary encoding of natural number \(n\). For example, this set contains \(0\$$1, \(1101\$$1100 and \(001\$$101\) but not \(1\$$1 or \(11\$$10.

You may use whichever form of acceptance – empty stack or final state – is convenient. In each case prove your machine is correct.

2. An NPDA

\[ M = (Q, \Sigma, \Gamma, \delta, s, \bot, F) \]

is a Binary-Stack NPDA if \(|\Gamma| = 2\). M is a Unary-Stack NPDA if \(|\Gamma| = 1\).

(a) Prove that every CFL is \(L_{es}(M)\) for some Binary-Stack NPDA \(M\).

\textbf{Hint:} You don’t need a grammar for this – you can do it entirely with machines – but think about the way a bottom-up recognizer as discussed in lecture implements “reduce” actions.
Give a language \( L \) that is not regular but is \( L_{es}(M) \) for some Unary-Stack NPDA \( M \).

Does there exist a CFL \( L \) that is not \( L_{es}(M) \) for any Unary-Stack NPDA \( M \)? Argue convincingly for your answer. A detailed proof is not necessary.

3. This is a “cumulative” problem – each part develops on the previous parts. We derive some properties of the Deterministic CFLs (DCFLs), i.e. languages that are \( L(M) \) for some Deterministic PDA \( M \).

(a) Show the set
\[
\{ a^i b^i c^i \mid i > 0 \}
\]
is not a CFL.

(b) Show the set
\[
\{ a^i b^j c^i \mid i, j > 0 \}
\]
is a DCFL.

(c) Show the DCFLs are not closed under intersection: give DCFLs \( L_1 \) and \( L_2 \) such that \( L_1 \cap L_2 \) is not a DCFL.

(d) In lecture and in the text we show that DCFLs are closed under complement:
\[
L \text{ is a DCFL} \implies \bar{L} \equiv (\Sigma^* - L) \text{ is a DCFL}
\]
It follows that the DCFLs cannot be closed under union (Why?). Give an example; that is, give two DCFLs \( L_1 \) and \( L_2 \) such that \( L_1 \cup L_2 \) is not a DCFL. (Note that \( L_1 \cup L_2 \) is certainly a CFL; it’s just not a deterministic one). Prove your answer.

(e) Let the tagged union of two languages be defined by
\[
L_0 \cup_t L_1 \equiv \{ 0w \mid w \in L_0 \} \cup \{ 1w \mid w \in L_1 \}
\]
Prove the DCFLs are closed under tagged union.

(f) Are the DCFLs closed under homomorphism? Explain your answer.