CS481F01 Homework 0

A. Demers

3 Sep 2001

This homework should help you decide which of CS381, CS481 is appropriate for you. It illustrates the style of mathematical (especially inductive) arguments that will be required in the rest of the course. The material covered here is not really part of the course (with the exception of some of the string results, which I think are intuitively obvious even if their proofs are not trivial), so this homework will count somewhat less than the remaining ones.

Preliminary Definitions

An *enumeration* of a set $\mathbf{S}$ is an onto function $f : \mathbb{N} \rightarrow \mathbf{S}$. Set $\mathbf{S}$ is *countable* if it has an enumeration.
$\Box$

An (irreflexive) *partial order* on $\mathbf{S}$ is a binary relation $\sqsubset$ satisfying

$$\begin{aligned}&\textbf{transitive} : (x \sqsubset y \wedge y \sqsubset z) \Rightarrow (x \sqsubset z)\\ &\textbf{irreflexive} : \neg(\exists x \;\; x \sqsubset x)\end{aligned}$$

$\Box$

An enumeration f of $\mathbf{S}$ is *consistent with* $\sqsubset$ if it satisfies

$$f(i) \sqsubset f(j) \quad \Rightarrow \quad i < j$$

for all i and j. That is, f enumerates $\mathbf{S}$ "in $\sqsubset$ order."
$\Box$

A partial order $\sqsubset$ on $\mathbf{s}$ is *well-founded* if

$$\forall x \in \mathbf{S} \;.\; \{y \in \mathbf{S} \;\; | y \sqsubset x\} \text{ is finite.}$$

In particular, in a well-founded relation there can be no infinite descending chains.
$\Box$

1: Facts about enumerations.

(a) Prove: if **S** has an enumeration consistent with $\sqsubset$ then $\sqsubset$ is well-founded.

(b) Prove: there is no enumeration of the rational numbers consistent with $<$, the usual arithmetic ordering.

(c) Prove: the rational numbers are countable.

(d) Let Σ be a finite alphabet with a total order defined on the symbols. Prove: there is no enumeration of Σ^* consistent with lexicographical order.

(e) Prove that Σ^* is countable.

2: A problem about strings. This problem might remind you of famous Euclid's famous GCD algorithm. Let Σ be a finite alphabet, and let $x, y \in \Sigma^*$. Prove that

$$(xy = yx) \quad \Leftrightarrow \quad \exists s \in \Sigma^*, i, j \in \mathbb{N} \; . \; (x = s^i \;\wedge\; y = s^j)$$

That is, s is a "factor" of both x and y.
$\square$

3: More infinite sets. An *arithmetic progression* over $\mathbb{N}$ is a set of the form

$$\mathcal{A}_{a,b} \;=\; \{\; a + ib \mid i \geq 0\}$$

where $a \geq 0$, $b > 0$.

Certainly there are subsets of $\mathbb{N}$ that intersect every arithmetic progression – for example, $\mathbb{N}$ itself is such a subset.

(a) Prove: no finite subset of $\mathbb{N}$ intersects every arithmetic progression.

(b) Prove there is a co-infinite subset of $\mathbb{N}$ intersects every arithmetic progression. (A co-infinite set is the complement of an infinite set; i.e., a set **S** such that $\mathbb{N} - \mathbf{S}$ is infinite).

(c) Does the answer to part (c) change if we weaken the definition of an arithmetic progression to allow $b \geq 0$ instead of $b > 0$?

(d) Show that all arithmetic progressions can be intersected by sets that are arbitrarily sparse in the following sense: for every function $f : \mathbb{N} \to \mathbb{N}$ there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that $g \geq f$ and $\mathbf{range}(g)$ intersects every arithmetic progression. That is, g enumerates a set that intersects every arithmetic progression and is more sparse than $\mathbf{range}(f)$.

Note part (d) implies part (b), but is a bit more difficult.
$\square$