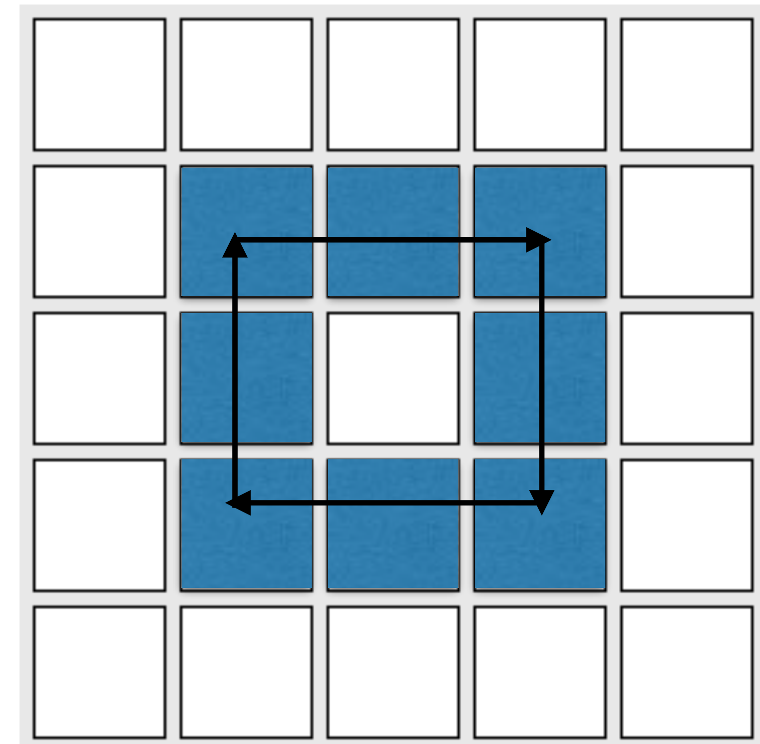
Machine Learning for Data Science (CS4786)
Lecture 20

Hidden Markov Models

Hidden Markov Model (HMM)

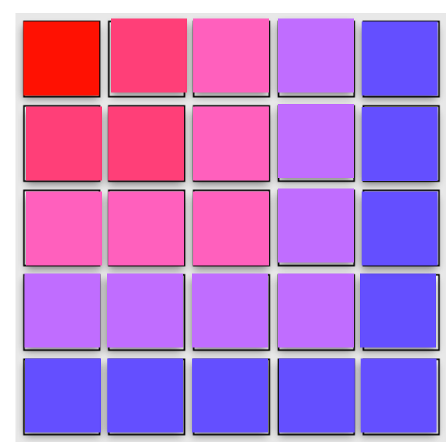
Same example:



But you don't observe location (dark room)

You hear how close the bot is!

What you hear:



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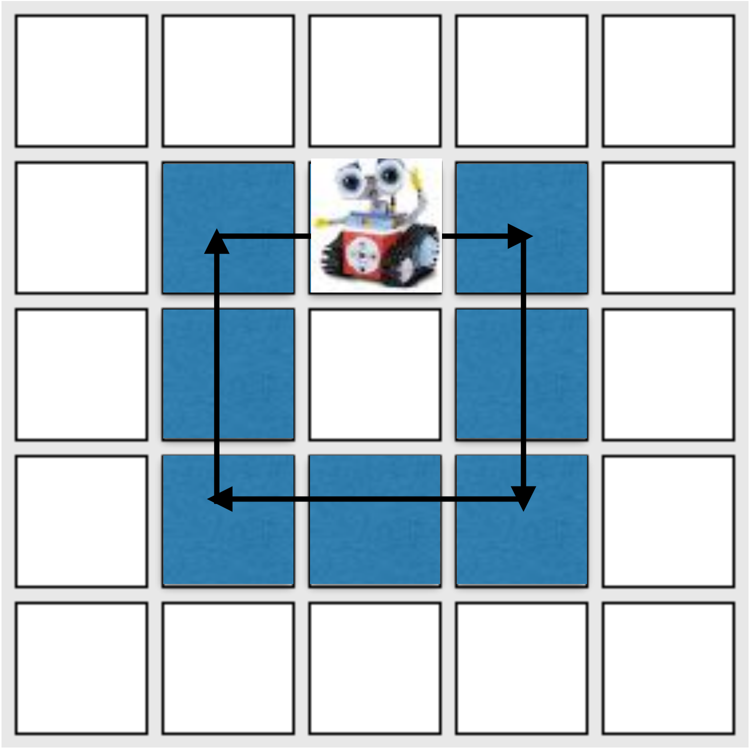

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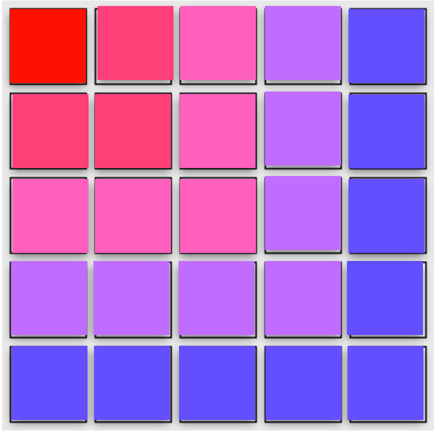


Can you catch the Bot?

HIDDEN MARKOV MODEL (HMM)

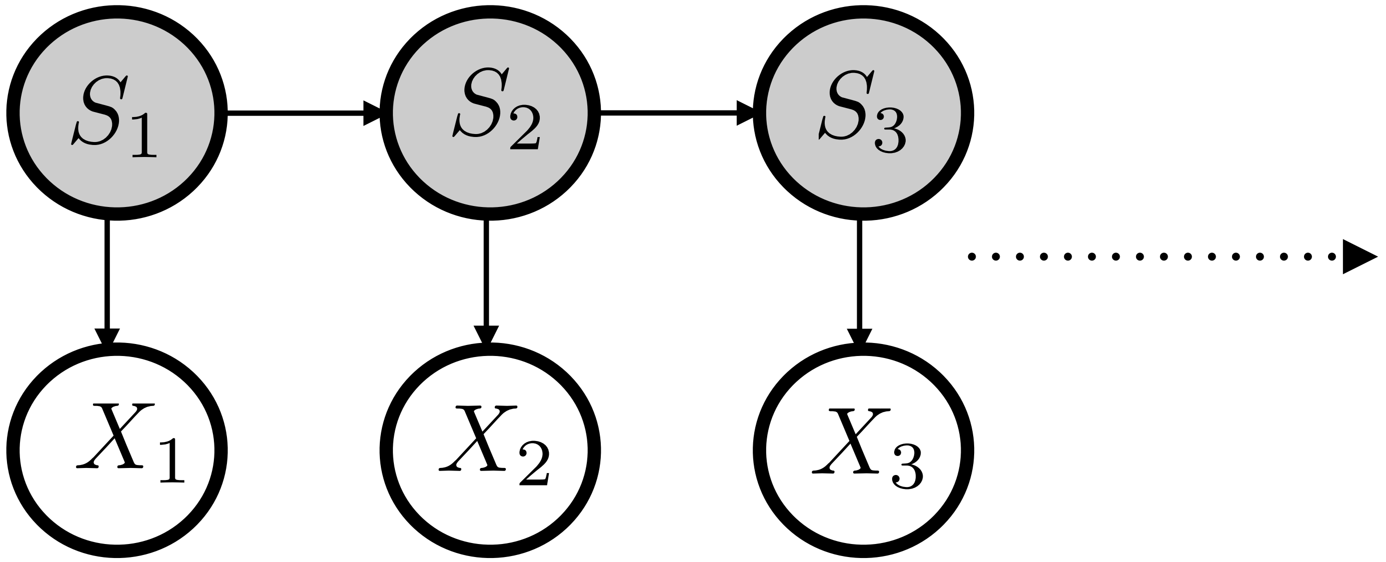
X_t's are what you hear (observation)

S_t's are the unseen locations (states)

Eg: for n x n grid we have, $K = n^2$ states

Number of alphabets = 5
(colors you can observe)

Hidden Markov Model (HMM)



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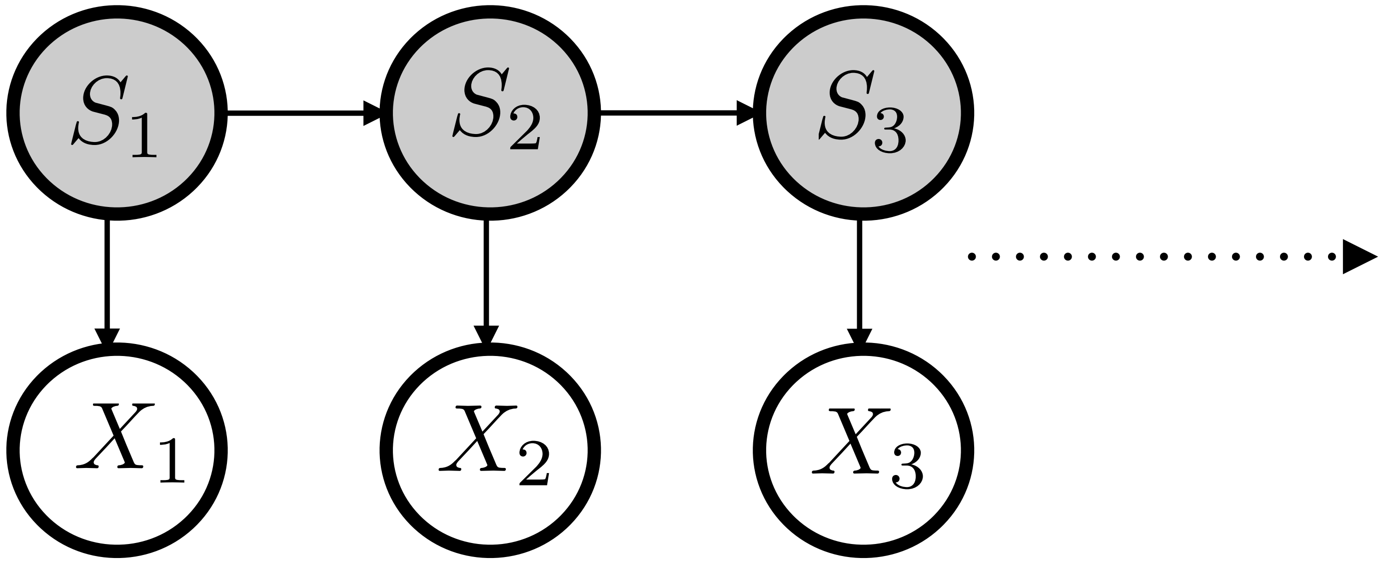

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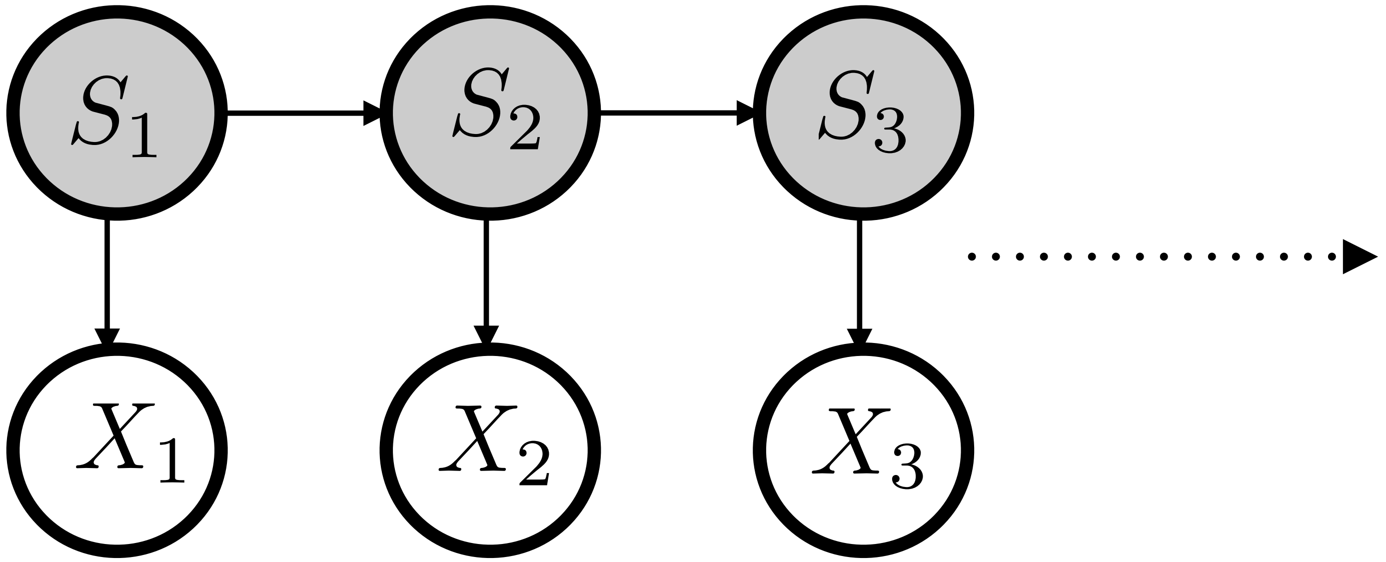
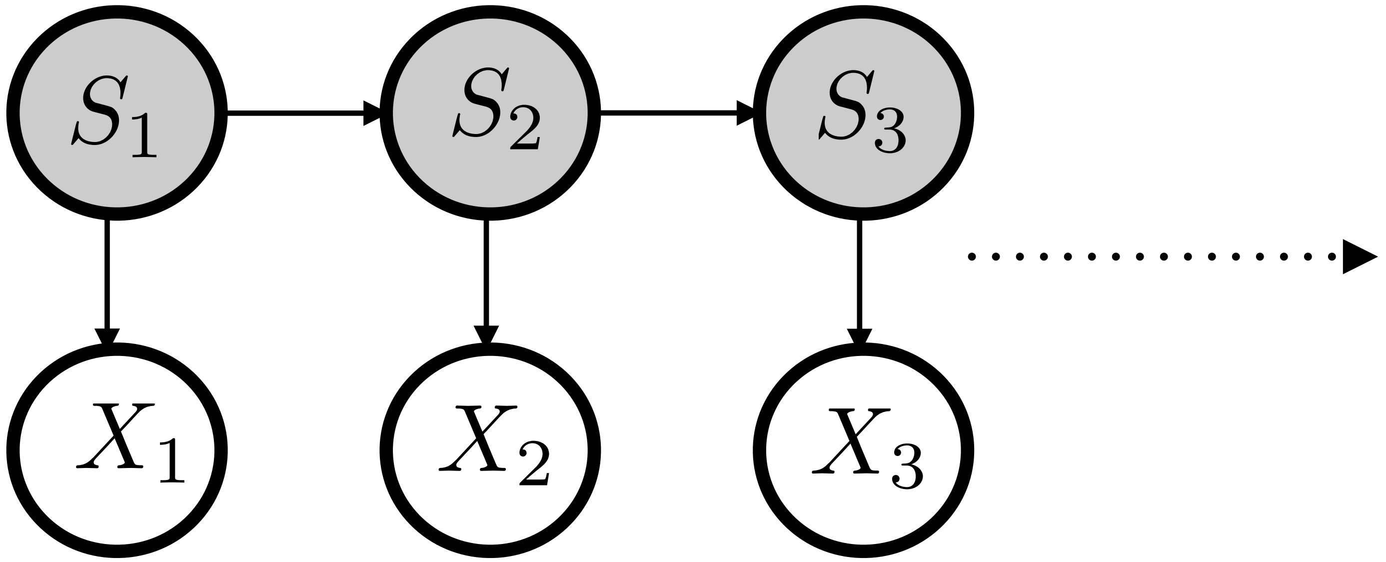


What are the parameters?

Hidden Markov Model (HMM)



What are the parameters?

Transition Probability table: T = P(S_t|S_{t-1})

Emission Probabilities: E = P(X_t|S_t)

Initial State Probabilities: P(S_1)

Hidden Markov Model (HMM)

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- What is probability that bot will be in location k at time t given the entire sequence of observations?

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$$P(S_t = k | X_1, \ldots, X_N)?$$

Inference in HMM

$P(S_t = k | X_1, \ldots, X_N)$

Inference in HMM

$$
\begin{aligned}
&P(S_t = k | X_1, \ldots, X_N) \\
&\qquad \propto P(X_{t+1}, \ldots, X_N | S_t = k, X_1, \ldots, X_t) P(S_t = k | X_1, \ldots, X_t)
\end{aligned}
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&\quad \propto P(X_{t+1}, \ldots, X_N|S_t = k)P(X_t|S_t = k)P(S_t = k, X_1, \ldots, X_{t-1})
\end{aligned}
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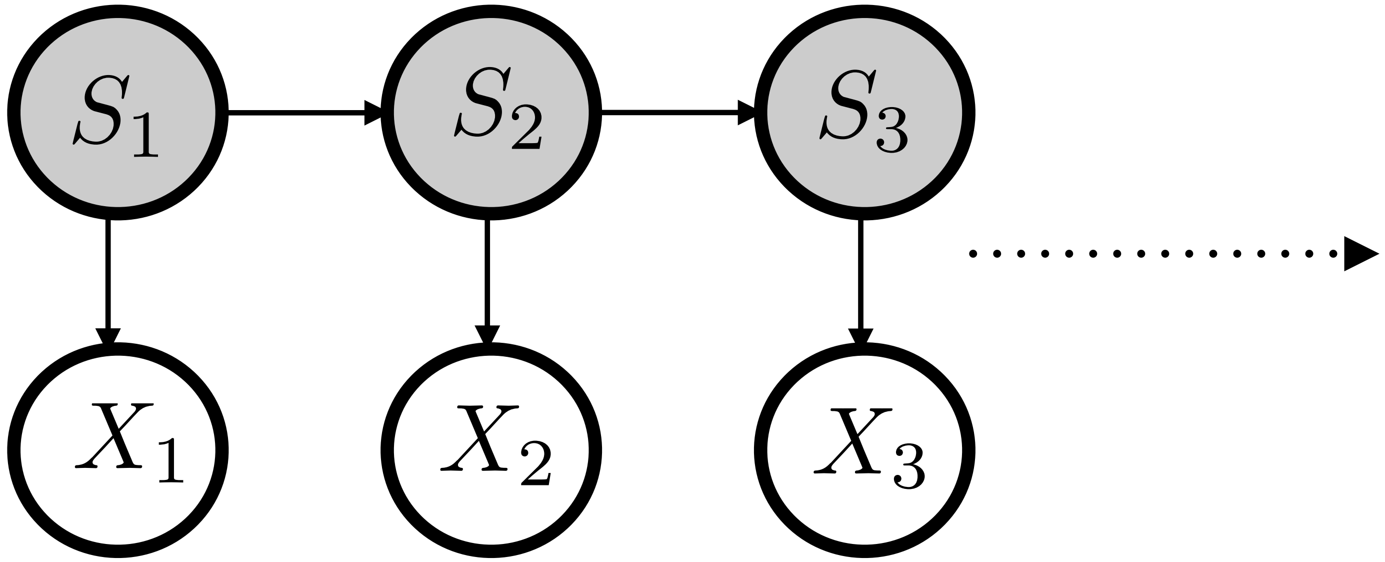
&\propto P(X_{t+1}, \ldots, X_N | S_t = k) P(X_t | S_t = k) P(S_t = k, X_1, \ldots, X_{t-1})
\end{aligned}
$$

We know $P(X_t | S_t = k)$'s and $P(S_t | S_{t-1})$

Compute $P(X_{t+1}, \ldots, X_N)$ and $P(S_t = k, X_1, \ldots, X_{t-1})$ recursively.

Inference in HMM

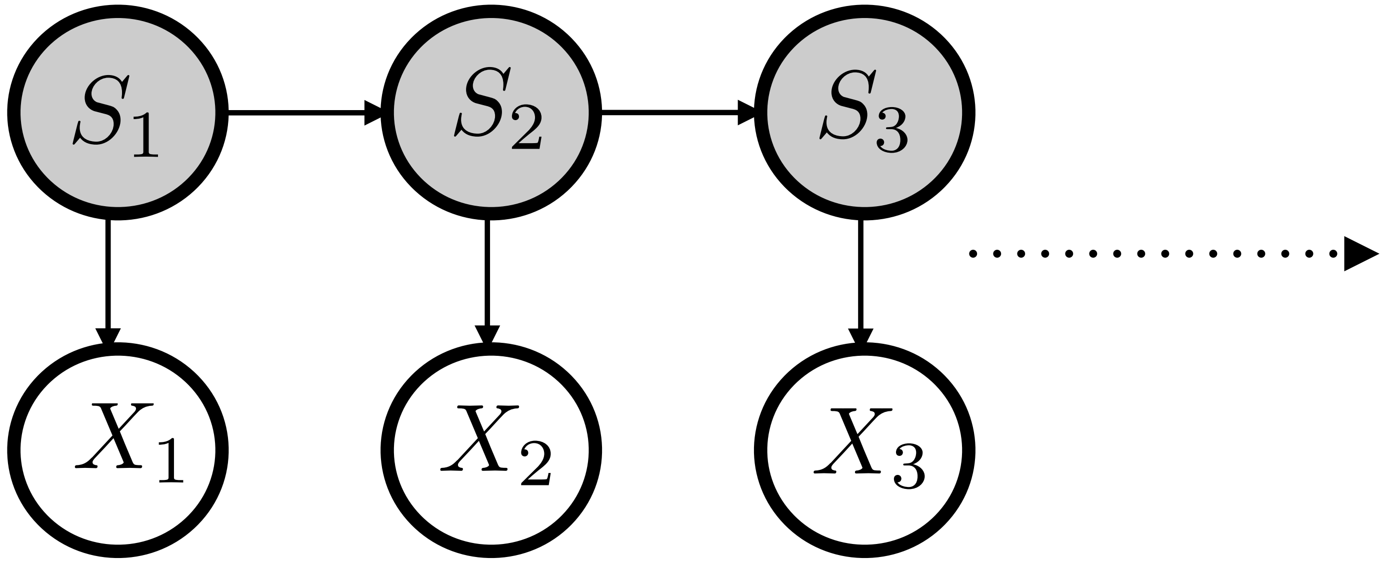


$$\text{message}_{S_{t-1} \mapsto S_t}(k) = P(S_t = k, X_1, \ldots, X_{t-1})$$

$$\text{message}_{S_{t+1} \mapsto S_t}(k) = P(X_n, \ldots, X_{t+1} | S_t = k)$$

$$P(S_t = k | X_1, \ldots, X_n) \propto \text{message}_{S_{t-1} \mapsto S_t}(k) \times \text{message}_{S_{t+1} \mapsto S_t}(k) \times P(X_t | S_t = k)$$

Inference in HMM



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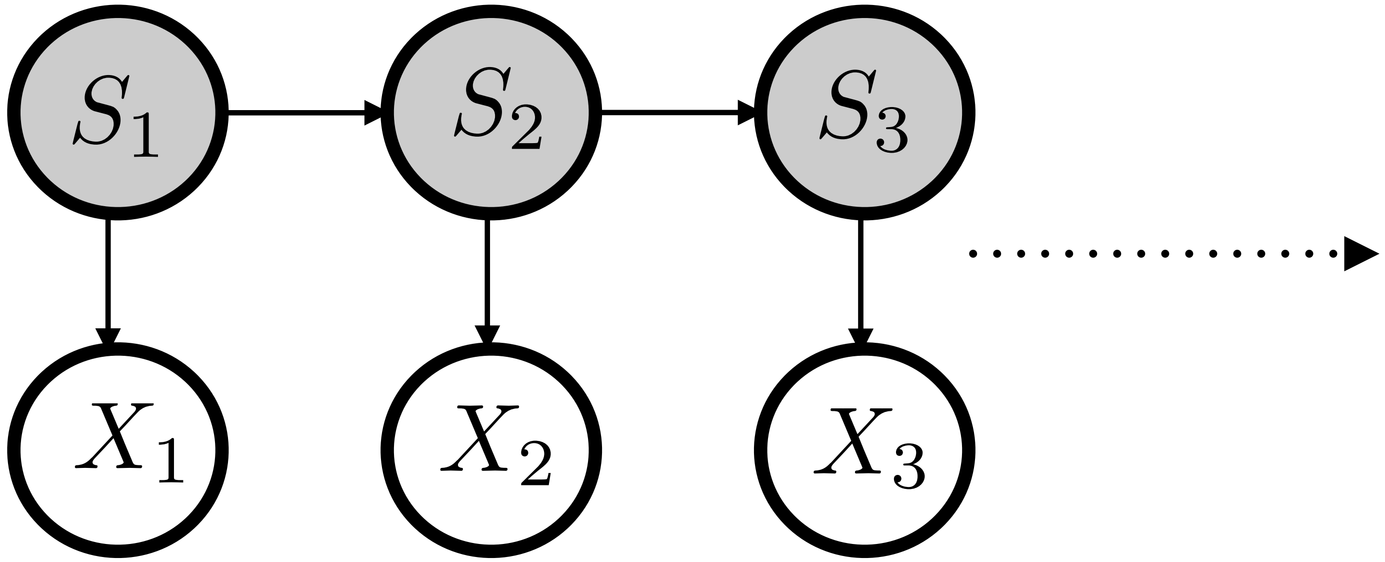

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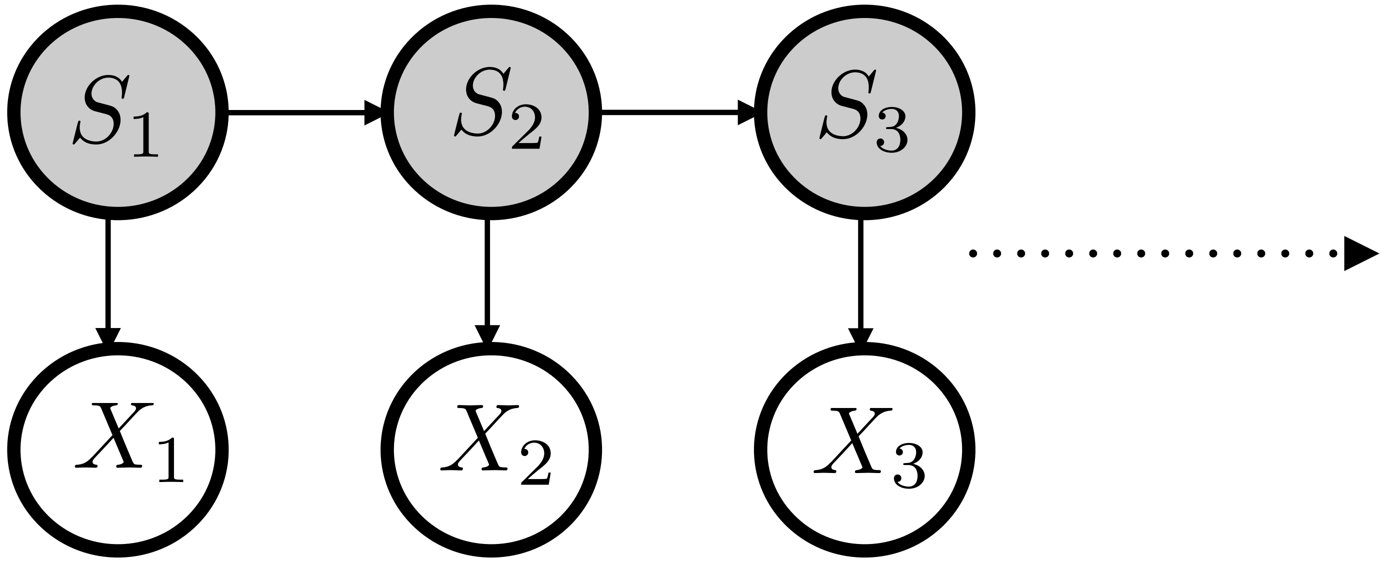
Forward:

$$P(X_1, \ldots, X_{t-1}, S_t = k) = \sum_{j=1}^{K} P(S_t = k | S_{t-1} = j) P(X_{t-1} | S_{t-1} = j) P(X_1, \ldots, X_{t-2}, S_{t-1} = j)$$

Inference in HMM



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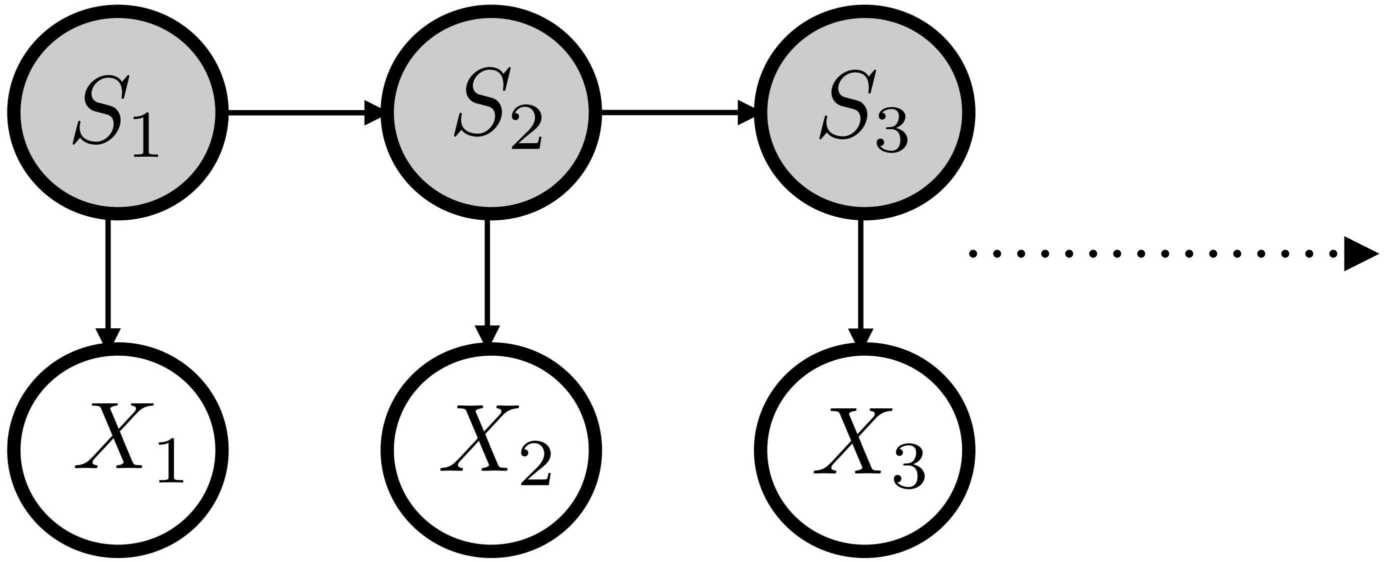
Inference in HMM



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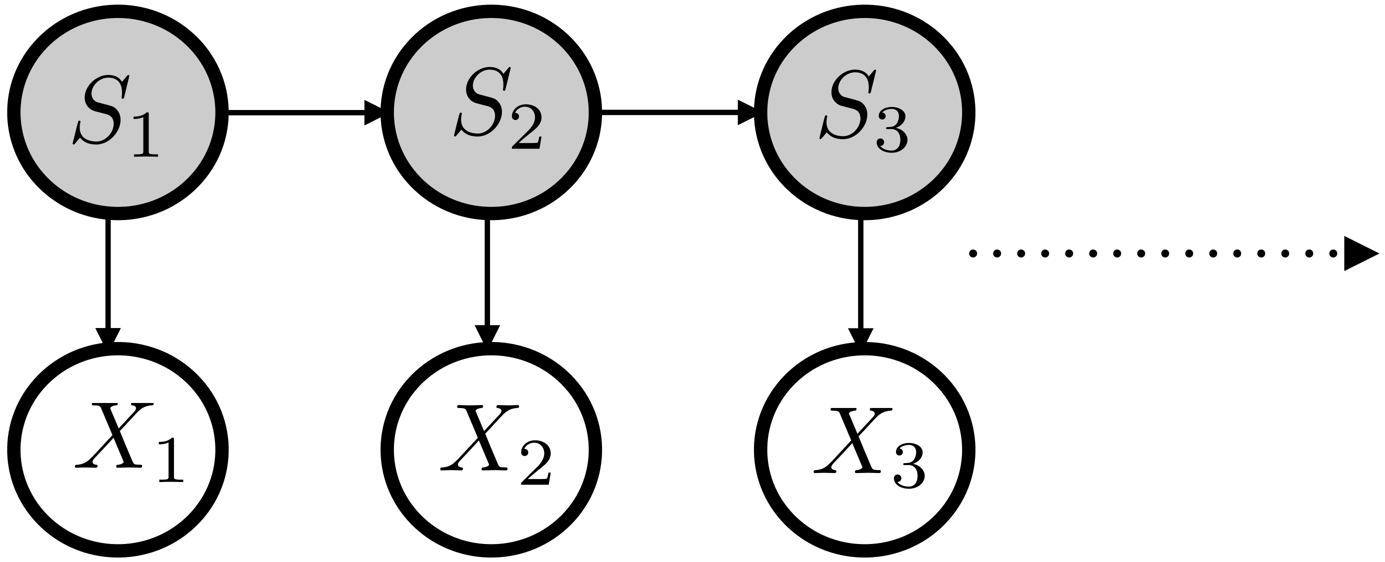

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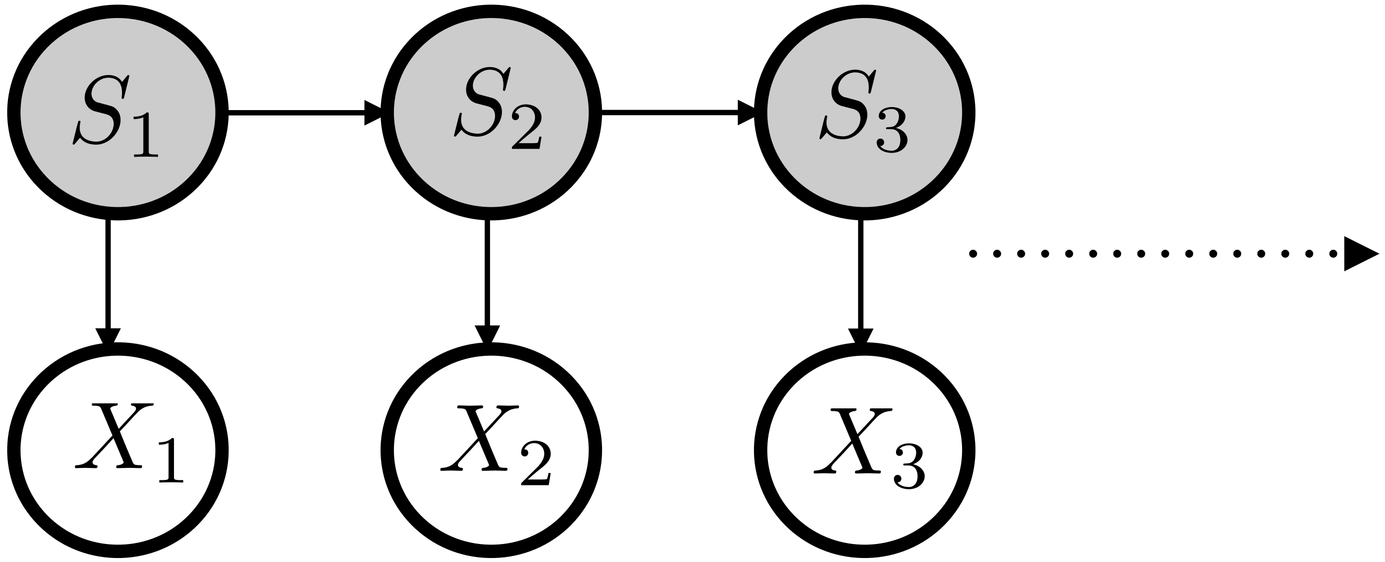
Backward:

$$P(X_n, \ldots, X_{t+1}|S_t = k) = \sum_{j=1}^{K} P(X_n, \ldots, X_{t+2}|S_{t+1} = j) P(X_{t+1}|S_{t+1} = j) P(S_{t+1} = j|S_t = k)$$

Inference in HMM



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Learning Parameters for HMM

- Now that we have algorithm for inference, what about learning
- Given observations, how do we estimate parameters for HMM? Three guesses …

EM for HMM (Baum Welch)

- EM algorithm of course, for HMM its referred to as Baum Welch algorithm
- Initialize Transition and Emission probability tables arbitrarily
- For $i = 1$ to convergence:

E-step For every state variable $t \in \{1, \ldots, n\}$,
Use forward-backward algorithm to compute probabilities of latent variables given obervation

M-step Optimize weighted log likelihood as usual:

$$\theta^{(i)} = \arg\max_{\theta \in \Theta} \sum_{S_{1,\ldots,n}} P(S_{1,\ldots,n} | X_{1,\ldots,n}, \theta^{(i-1)}) \log P(X_{1,\ldots,n}, S_{1,\ldots,n} | \theta)$$

Lets Simplify M-step

$$\log P(X_{1,\ldots,n}, S_{1,\ldots,n}|\theta)$$

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$$\log P(X_{1,\dots,n}, S_{1,\dots,n}|\theta) = \log\left(\prod_{t=1}^{n} P(X_t|S_t,\theta)\prod_{t=1}^{n} P(S_t|S_{t-1},\theta)\right)$$

:

Lets Simplify M-step

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$$= \sum_{t=1}^{n}\log P(X_t|S_t,\theta) + \sum_{t=1}^{n}\log P(S_t|S_{t-1},\theta)$$

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$$= \sum_{t=1}^{n}\log P(X_t|S_t,\theta) + \sum_{t=1}^{n}\log P(S_t|S_{t-1},\theta)$$

Hence,

$$\sum_{S_{1,\dots,n}} P(S_{1,\dots,n}|X_{1,\dots,n},\theta^{(i-1)})\log P(X_{1,\dots,n},S_{1,\dots,n}|\theta)$$

$$= \sum_{t=1}^{n}\sum_{s_t=1}^{K} P(S_t = s_t|X_{1,\dots,n},\theta^{i-1})\log P(X_t|S_t = s_t,\theta)$$

$$+ \sum_{t=1}^{n}\sum_{s_t,s_{t-1}=1}^{K} P(S_t = s_t, S_{t-1} = s_{t-1}|X_{1,\dots,n},\theta^{i-1})\log P(S_t|S_{t-1},\theta)$$

E-step

E-STEP

- Only need to compute $P(S_t = s_t | X_{1,\ldots,n}, \theta^{i-1})$ and $P(S_t = s_t, S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})$ using forward-backward

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- First term is immediate

$$P(S_t = s_t | X_{1,\ldots,n}, \theta^{i-1}) \propto m_{S_{t-1} \mapsto S_t}(s_t) \cdot m_{S_{t+1} \mapsto S_t}(s_t) \cdot E^{(i-1)}[s_t, X_t]$$

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- For second term,

$$P(S_t = s_i, S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})$$
$$\propto m_{S_{t-1} \mapsto S_t}(s_t) T^{(i-1)}[s_{t-1}, s_t] P(S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})$$
$$\propto m_{S_{t-1} \mapsto S_t}(s_t) T^{(i-1)}[s_{t-1}, s_t] m_{S_{t-2} \mapsto S_{t-1}}(s_{t-1}) m_{S_t \mapsto S_{t-1}}(s_{t-1}) E^{(i-1)}[s_{t-1}, X_{t-1}]$$

Why?

E-step

$$P(S_t = s_t, S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})$$

E-STEP

$$P(S_t = s_t, S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})$$

$$= P(S_t = s_t, | S_{t-1} = s_{t-1}, X_{1,\ldots,n}, \theta^{t-1}) P(S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})$$

E-step

$$P(S_t = s_t, S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})$$
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\begin{aligned}
&P(S_t = s_t, S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1}) \\
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&\quad \propto P(X_{t,\ldots,n} | S_t = s_t, S_{t-1} = s_{t-1}, \theta^{i-1}) \\
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&P(S_t = s_t, S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1}) \\
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&\quad \propto P(X_{t,\ldots,n} | S_t = s_t, S_{t-1} = s_{t-1}, \theta^{i-1}) \\
&\qquad\qquad P(S_t = s_t | S_{t-1} = s_{t-1}, \theta^{i-1}) P(S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1}) \\
&\quad \propto P(X_{t,\ldots,n} | S_t = s_t, \theta^{i-1}) \\
&\qquad\qquad T^{(i-1)}[s_{t-1}, s_t] P(S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})
\end{aligned}
$$

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&P(S_t = s_t, S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1}) \\
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&\quad \propto m_{S_{t-1} \mapsto S_t}(s_t) \cdot T^{(i-1)}[s_{t-1}, s_t] \cdot P(S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})
\end{aligned}
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& \quad \propto P(X_{t,\ldots,n} | S_t = s_t, S_{t-1} = s_{t-1}, \theta^{i-1}) \\
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& \quad \propto m_{S_{t-1} \mapsto S_t}(s_t) \cdot T^{(i-1)}[s_{t-1}, s_t] \cdot P(S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1}) \\
& \quad \propto m_{S_{t-1} \mapsto S_t}(s_t) \cdot T^{(i-1)}[s_{t-1}, s_t] \\
& \qquad\qquad m_{S_{t-2} \mapsto S_{t-1}}(s_{t-1}) \cdot m_{S_t \mapsto S_{t-1}}(s_{t-1}) \cdot E^{(i-1)}[s_{t-1}, X_{t-1}]
\end{aligned}
$$

Baum Welch Algorithm

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Initialize T^0, E^0 probability tables

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For $i = 1$ to convergence

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- E-step:
  - Run Forward-Backward algorithm and compute messages

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For $i = 1$ to convergence

- E-step:
  - Run Forward-Backward algorithm and compute messages
  - For every t compute $P(S_t = s_t, S_{t-1} = s_{t-1}|X_{1,\ldots,n}, \theta^{i-1})$ and $P(S_t = s_t|X_{1,\ldots,n}, \theta^{i-1})$ as in previous slides

Baum Welch Algorithm

Initialize T^0, E^0 probability tables

For $i = 1$ to convergence

- E-step:
  - Run Forward-Backward algorithm and compute messages
  - For every t compute $P(S_t = s_t, S_{t-1} = s_{t-1} | X_{1,\ldots,n}, \theta^{i-1})$ and $P(S_t = s_t | X_{1,\ldots,n}, \theta^{i-1})$ as in previous slides
- M-step:

$$\forall u, v \quad T^{(i)}[u, v] = \frac{\sum_{t=2}^{n} P(S_t = v, S_{t-1} = u | X_{1,\ldots,n}, \theta^{i-1})}{\sum_{t=2}^{n} P(S_{t-1} = u | X_{1,\ldots,n}, \theta^{i-1})}$$

$$\forall v, e \quad E^{(i)}[v, e] = \frac{\sum_{t=1}^{n} P(S_t = v | X_{1,\ldots,n}, \theta^{i-1}) \cdot \mathbf{1}_{X_t = e}}{\sum_{t=1}^{n} P(S_t = v | X_{1,\ldots,n}, \theta^{i-1})}$$

General Bayesian Network

Bayesian Networks

- Directed acyclic graph (DAG): $G = (V, E)$
- Joint distribution P_θ over $X_1, \ldots, X_n$ that factorizes over G:

$$P_\theta(X_1, \ldots, X_n) = \prod_{i=1}^{N} P_\theta(X_i | \text{Parent}(X_i))$$

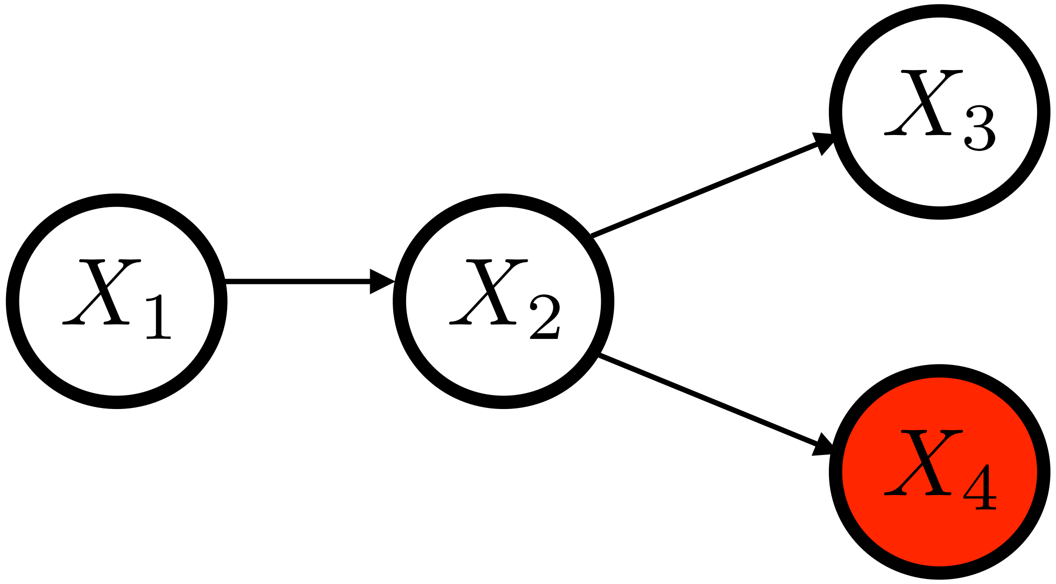
- Hence Bayesian Networks are specified by G along with CPD's over the variables (given their parents)

Variable Elimination: Examples

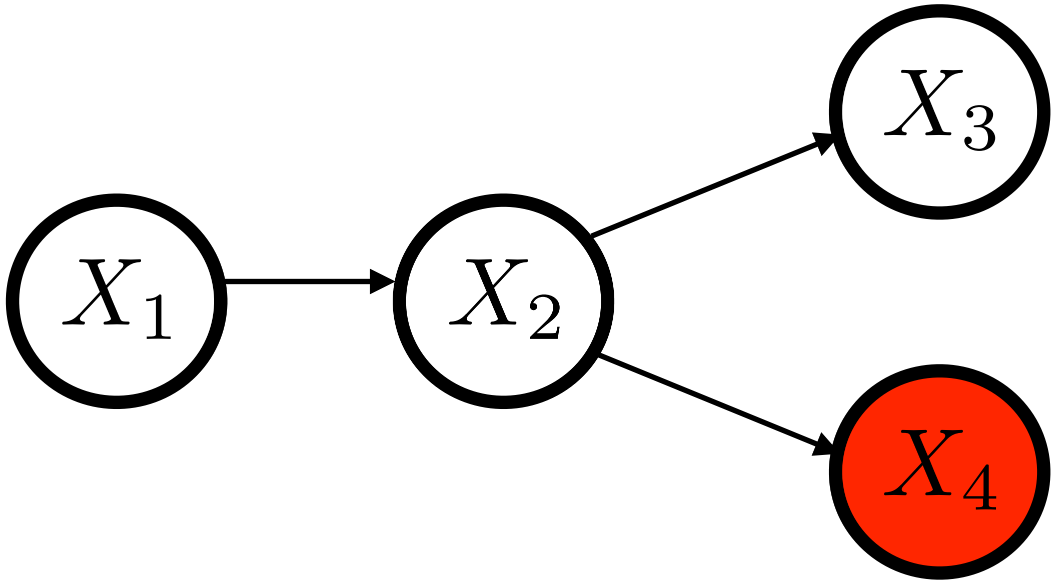
- Marginals are enough:

$$P(X_j = x_j, X_k = x_k | X_i = x_i, X_h = x_h) = \frac{P(X_j = x_j, X_k = x_k, X_i = x_i, X_h = x_h)}{P(X_i = x_i, X_h = x_h)}$$

Variable Elimination: Examples

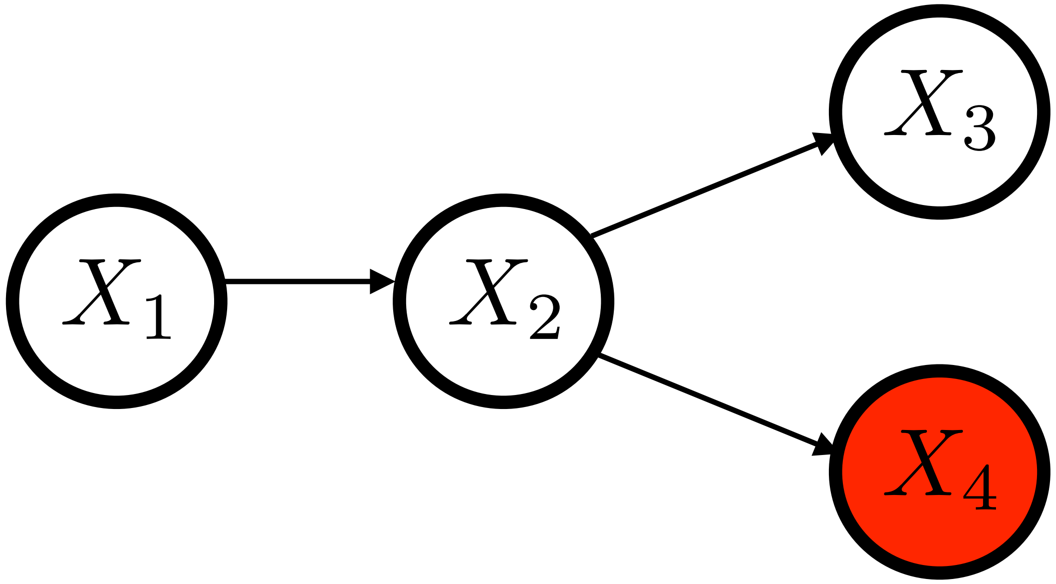


Variable Elimination: Examples



$$P(X_4) = \sum_{x_1} \sum_{x_2} \sum_{x_3} P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4)$$

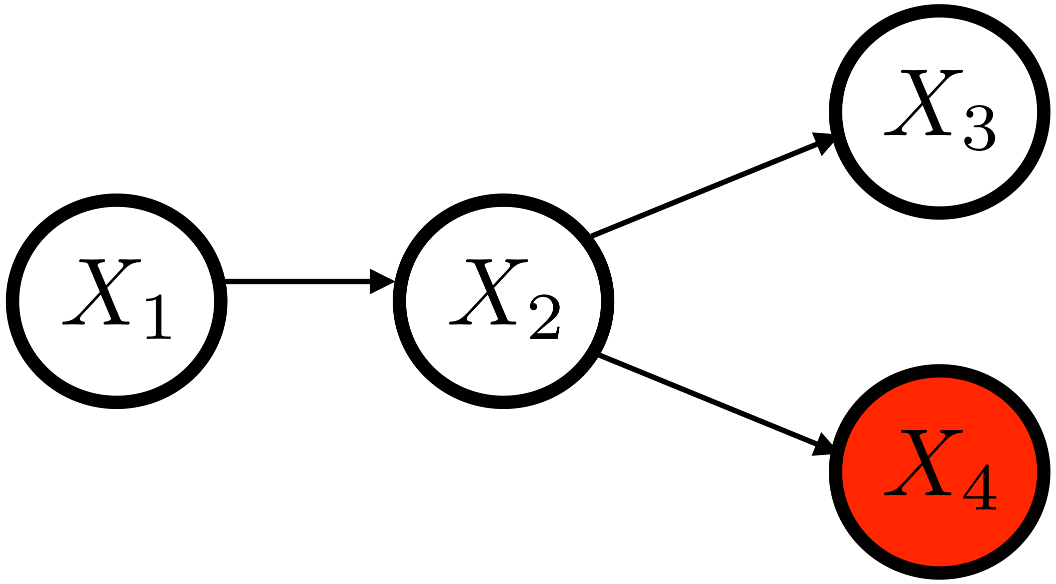
Variable Elimination: Examples



$$P(X_4) = \sum_{x_1}\sum_{x_2}\sum_{x_3} P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4)$$

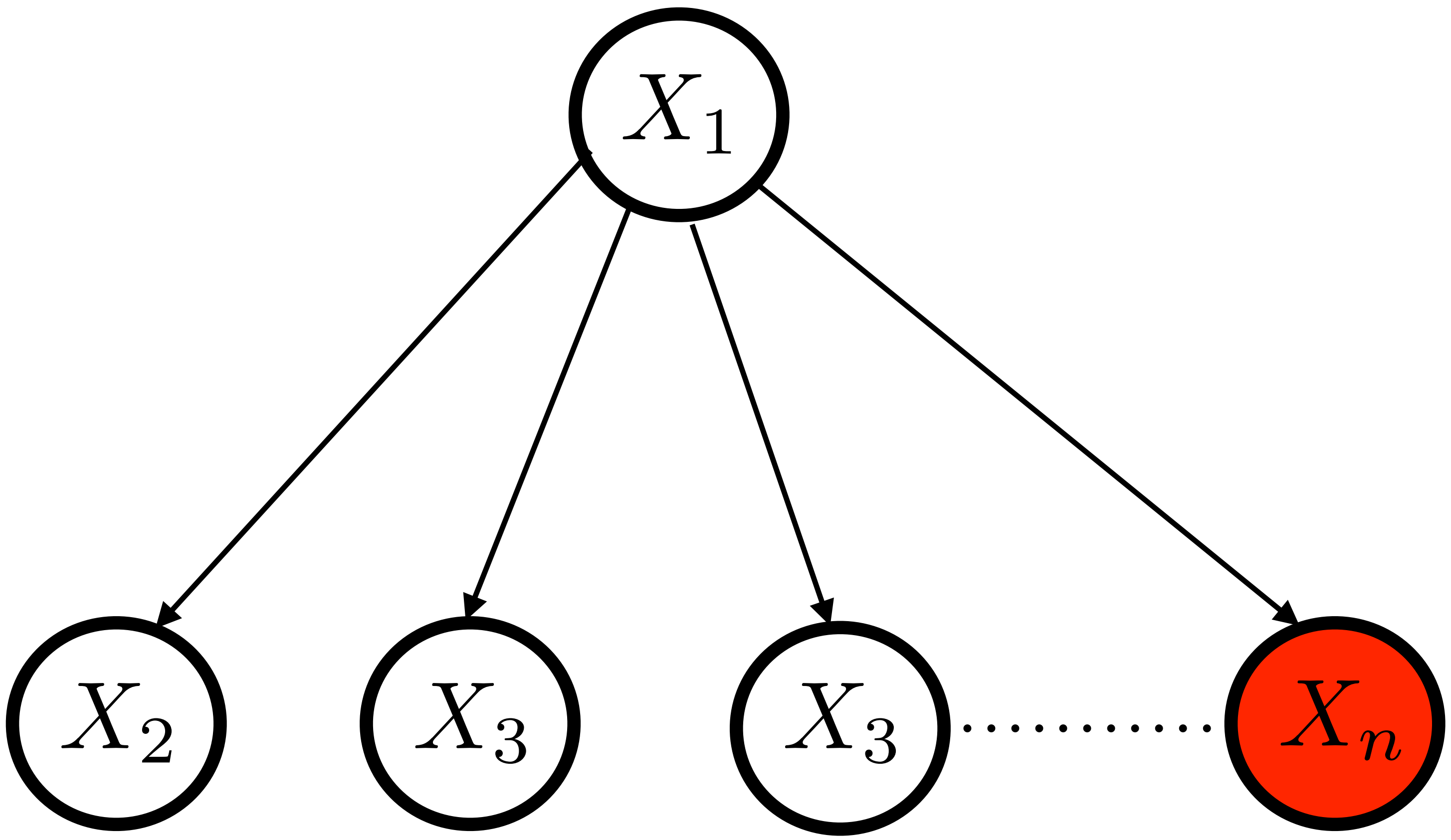
$$= \sum_{x_1}\left(P(X_1 = x_1)\sum_{x_2}\left(P(X_2 = x_2|X_1 = x_1)P(X_4|X_2 = x_2)\left(\sum_{x_3} P(X_3 = x_3|X_2 = x_2)\right)\right)\right)$$

Variable Elimination: Examples



$$
\begin{aligned}
P(X_4) &= \sum_{x_1}\sum_{x_2}\sum_{x_3} P(X_1 = x_1, X_2 = x_2, X_3 = x_3, X_4) \\
&= \sum_{x_1}\left(P(X_1 = x_1)\sum_{x_2}\left(P(X_2 = x_2|X_1 = x_1)P(X_4|X_2 = x_2)\left(\sum_{x_3} P(X_3 = x_3|X_2 = x_2)\right)\right)\right) \\
&= \sum_{x_1}\left(P(X_1 = x_1)\left(\sum_{x_2} P(X_2 = x_2|X_1 = x_1)P(X_4|X_2 = x_2)\right)\right)
\end{aligned}
$$

Variable Elimination: Order Matters



Right order: O(n)

Wrong order: $O(2^n)$