Gaussian Mixture Models

Question 1. Say you are given location of n points, $x_1, \ldots, x_n$. However you are color blind and you are only able to get some estimate probability each of the points being of color red, blue or green. That is, say $Q_t[k]$ be the probability of the t'th point being of color k. (So we have $\sum_k Q_t[k] = 1$). Now given this, write down:

1. Estimate for number of balls of color k
2. Estimate for the mean μ_k of all the points of color k
3. Estimate for the covariance Σ_k of all the points of color k

Question 2. To make things simple lets just take a one dimensional example. Say we are given n numbers $x_1, \ldots, x_n$ and we are given $Q_1, \ldots, Q_n$ where $Q_t[k]$ is the probability of the t'th number being in cluster k. You are interested in the following optimization for the parameters of the Gaussian Mixture model over these points:

$$\begin{aligned}
&\text{argmax}_{\theta=(\pi,\mu_1,\ldots,\mu_k,\sigma_1,\ldots,\sigma_K)} \sum_{t=1}^{n}\sum_{k=1}^{K} Q_t[k] \log P(x_t, c_t = k|\theta) \\
&= \text{argmax}_{\theta=(\pi,\mu_1,\ldots,\mu_k,\sigma_1,\ldots,\sigma_K)} \sum_{t=1}^{n}\sum_{k=1}^{K} Q_t[k] \left(\log \phi(x_t; c_t = k; \mu_k, \sigma_k) + \log \pi_k\right)
\end{aligned}$$

Find the solution for the above optimization, that is write down π, μ's and σ's. (Hint: for π since its mixture probability it needs to sum to 1 and has to be non-negative.)