1 t-SNE

Question 1: Given points $\mathbf{x}_1, \ldots, \mathbf{x}_n$, for any $t, s \in [n]$, let

$$p_{t \to s} = \frac{\exp(-\frac{\|\mathbf{x}_s - \mathbf{x}_t\|^2}{2\sigma^2})}{\sum_{u \neq t} \exp(-\frac{\|\mathbf{x}_u - \mathbf{x}_t\|^2}{2\sigma^2})}$$

Now define $P_{s,t} = \frac{p_{t \to s} + p_{s \to t}}{2n}$ and assume $P_{t,t} = 0$ for any t. Show that P is a valid probability distribution over $[n] \times [n]$.

2 Spectral Embedding

Consider an undirected graph $G = (V, E)$ consisting of $|V| = n$ vertices and E is the set of edges that indicate which pairs of vertices are connected. Let A be the $n \times n$ adjacency matrix of the graph where $A[i, j] = 1$ if node i and node j have an edge between them. Let D be the $n \times n$ diagonal matrix with entry $D[i, i]$ indicating the degree of node i in the graph.

Question 2: Let y be any n dimensional vector (one coordinate for every node in the graph). Show that

$$\sum_{(u,v)\in E} (\mathbf{y}_u - \mathbf{y}_v)^2 = \mathbf{y}^\top (D - A)\mathbf{y}$$

(Assume in the above sum we only count each edge in only one direction)