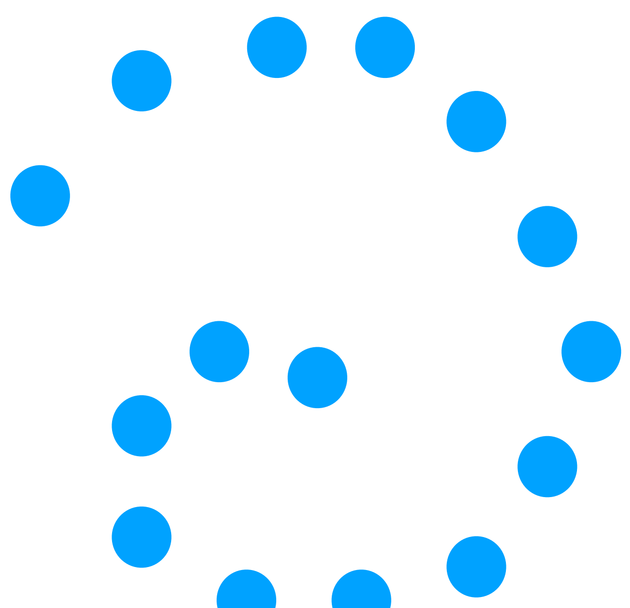
Machine Learning for Data Science (CS4786)
Lecture 9

Isomap + TSNE

Manifold Based Dimensionality Reduction

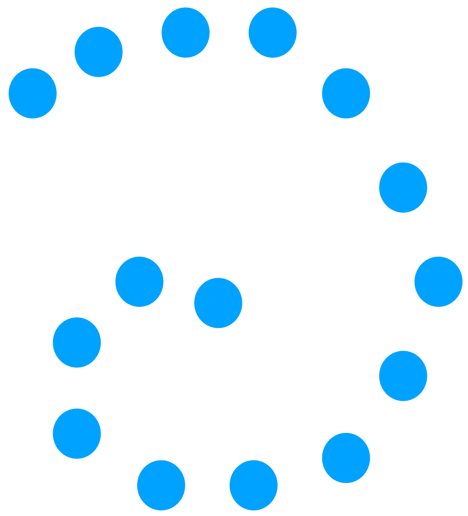
- Key Assumption: Points live on a low dimensional manifold
- Manifold: subspace that looks locally Euclidean
- Given data, can we uncover this manifold?



Can we unfold this?

Method I: Isomap

1. For every point, find its (k-) Nearest Neighbors



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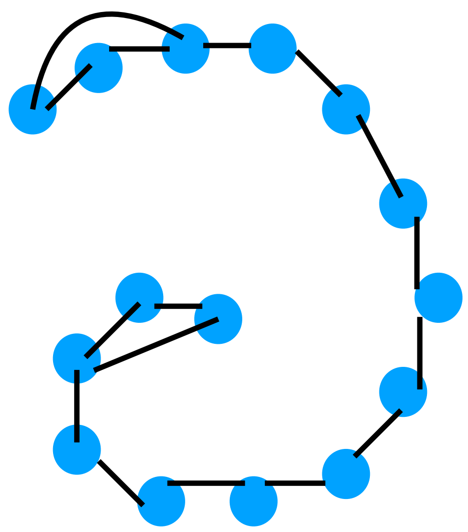

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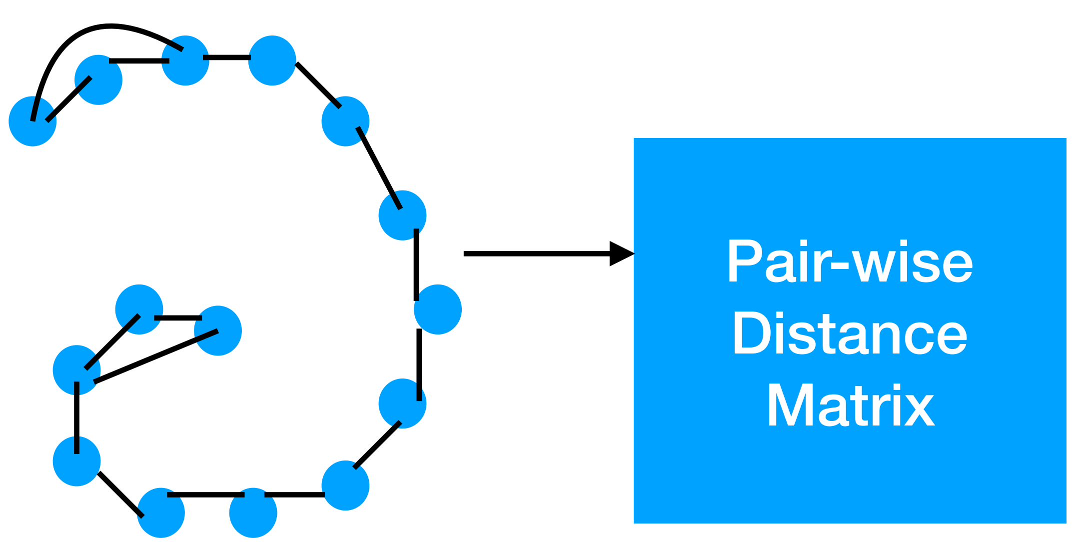

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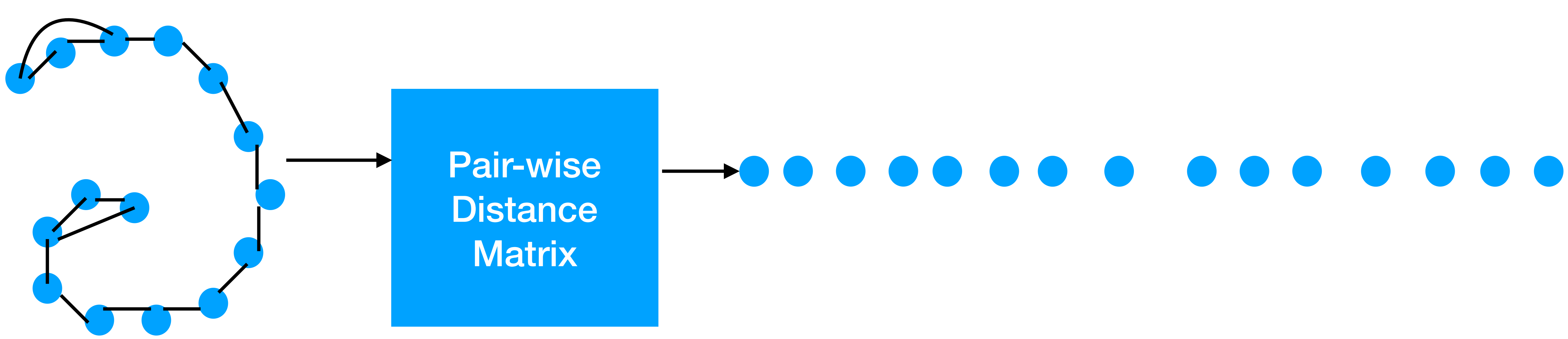
4. Find points in low dimensional space such that distances between points in this space is equal to distance on graph.



Isomap: Pitfalls

1. If we don't take enough nearest neighbors, then graph may not be connected
2. If we connect points too far away, points that should not be connected can get connected
3. There may not be a right number of nearest neighbors we should consider!

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  Probability that points s and t are connected $P_{s,t} = P_{t,s} = \frac{p_{t \to s} + p_{s \to t}}{2n}$

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  i.e. minimize:

$$\mathrm{KL}(P\|Q) = \sum_{s,t} P_{s,t}\log\left(\frac{P_{s,t}}{Q_{s,t}}\right) = \sum_{s,t} P_{s,t}\log\left(P_{s,t}\right) - \sum_{s,t} P_{s,t}\log\left(Q_{s,t}\right)$$

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  - For d dimensional gaussians, most points are found at distance $\sqrt{d}$ from mean!
  - If we use gaussians in both high and low dimensional space, all the points are squished in to a small space
  - Too many points crowd the center!

Method II: t-SNE

- Instead for Q we use, student t distribution which is heavy tailed:

$$q_{t \to s} = \frac{\left(1 + \|\mathbf{y}_s - \mathbf{y}_t\|^2\right)^{-1}}{\sum_{u \neq t} \left(1 + \|\mathbf{y}_u - \mathbf{y}_t\|^2\right)^{-1}}$$

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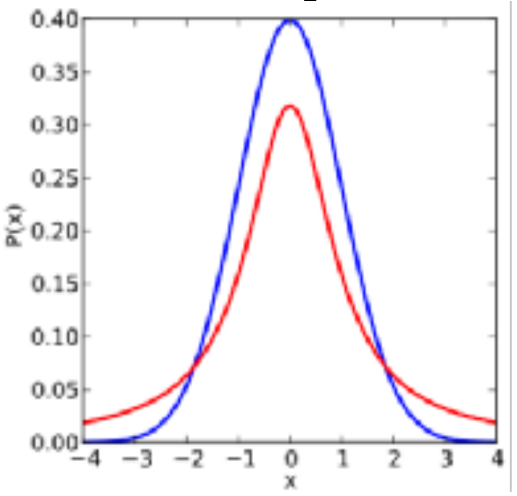

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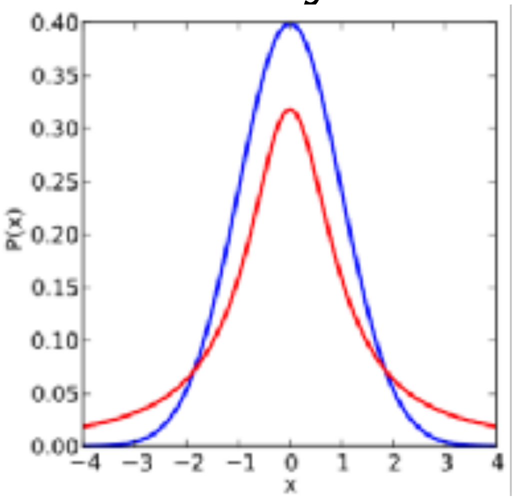

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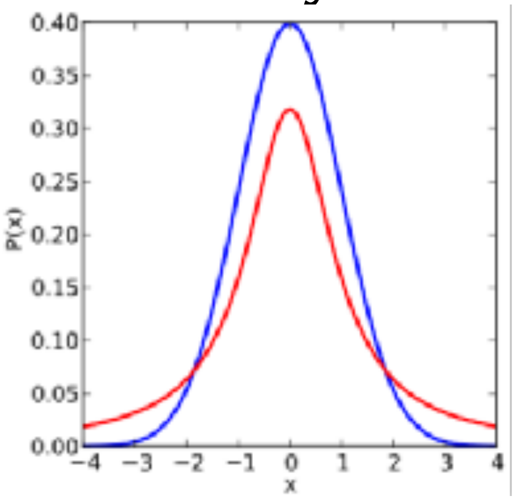

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- Algorithm: Find $\mathbf{y}_1, \ldots, \mathbf{y}_n$ by performing gradient descent

Demo