1 PCA Handout II

Assume the vectors $w_1, \ldots, w_d$ are all unit length vectors, that is,

$$\forall i \in [d], \quad \|w_i\|^2 = \sum_{j=1}^{d} w_i[j]^2 = 1$$

and are such that for any $i \neq j$, $w_i \perp w_j$, that is:

$$\sum_{k=1}^{d} w_i[k] \cdot w_j[k] = 0$$

It is a fact from Linear algebra that any vector in $d$ dimensions can be written as a linear combination of the orthonormal basis vectors: $w_1, \ldots, w_d$. Now say the vector $x_t - \mu$ is written as a linear combination of the basis vectors as:

$$x_t = \mu + \sum_{j=1}^{d} y_t[j]w_j$$

and define the vector $\hat{x}_t$ by taking $x_t - \mu$ as linear combination of only first $K$ of the basis, that is:

$$\hat{x}_t = \mu + \sum_{j=1}^{K} y_t[j]w_j$$

**Question 1:** Show that,

$$\text{dist}^2(\hat{x}_t, x_t) = \|\hat{x}_t - x_t\|^2 = \sum_{j=K+1}^{d} y_t[j]^2$$
Question 2: Say for any $K$, we can show that:

$$
\frac{1}{n} \sum_{t=1}^{n} \|\hat{x}_t - x_t\|_2^2 = \sum_{j=K+1}^{d} w_j^\top \Sigma w_j
$$

Then show that

$$
\sum_{j=1}^{d} w_j^\top \Sigma w_j = \frac{1}{n} \sum_{t=1}^{n} \|x_t - \mu\|_2^2
$$