1 PCA Handout II

Assume the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_d$ are all unit length vectors, that is,

$$\forall i \in [d], \quad \|\mathbf{w}_i\|_2^2 = \sum_{j=1}^{d} \mathbf{w}_i[j]^2 = 1$$

and are such that for any $i \neq j$, $\mathbf{w}_i \perp \mathbf{w}_j$, that is:

$$\sum_{k=1}^{d} \mathbf{w}_i[k] \cdot \mathbf{w}_j[k] = 0$$

It is a fact from Linear algebra that any vector in d dimensions can be written as a linear combination of the orthonormal basis vectors: $\mathbf{w}_1, \ldots, \mathbf{w}_d$. Now say the vector $\mathbf{x}_t - \mu$ is written as a linear combination of the basis vectors as:

$$\mathbf{x}_t = \mu + \sum_{j=1}^{d} \mathbf{y}_t[j]\mathbf{w}_j$$

and define the vector $\hat{\mathbf{x}}_t$ by taking $\hat{\mathbf{x}}_t - \mu$ as linear combination of only first K of the basis, that is:

$$\hat{\mathbf{x}}_t = \mu + \sum_{j=1}^{K} \mathbf{y}_t[j]\mathbf{w}_j$$

Question 1: Show that,

$$\text{dist}^2(\hat{\mathbf{x}}_t, \mathbf{x}_t) = \|\hat{\mathbf{x}}_t - \mathbf{x}_t\|_2^2 = \sum_{j=K+1}^{d} \mathbf{y}_t[j]^2$$

Question 2: Say for any K, we can show that:

$$\frac{1}{n}\sum_{t=1}^{n}\|\hat{\mathbf{x}}_t - \mathbf{x}_t\|_2^2 = \sum_{j=K+1}^{d}\mathbf{w}_j^\top \Sigma \mathbf{w}_j$$

Then show that

$$\sum_{j=1}^{d}\mathbf{w}_j^\top \Sigma \mathbf{w}_j = \frac{1}{n}\sum_{t=1}^{n}\|\mathbf{x}_t - \mu\|_2^2$$