Machine Learning for Data Science (CS4786)
Lecture 15

Probabilistic Modelling, MLE Vs MAP Vs Bayesian,
Latent Variables

Mar 19, 2015

Course Webpage :
http://www.cs.cornell.edu/Courses/cs4786/2015sp/

Clustering

- For arbitrary set of points, we can have either
  - Scale invariance
  - Consistency

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- Assume we have set Θ of possible models and data is generated from one of these $\theta \in \Theta$:

$$(x_t, c_t) \sim P_\theta(|(x_1, c_1), \ldots, (x_{t-1}, c_{t-1}))$$

EXAMPLES

- Apple doesn't fall far from its tree model:
  - Each θ consists of position of initial trees $\mu_1, \ldots, \mu_K \in \mathbb{R}^2$ and mixture distribution $\pi = (\pi_1, \ldots, \pi_K)$ where π_i is the probability with which we get tree of fruit i
  - At time t we generate a new tree as follows:
    - $c_t \sim \pi$
    - $\text{Parent}_t \sim$ pick a parent tree uniformly from one of the c_t trees
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- Gaussian Mixture Model
  - Each θ consists of mixture distribution $\pi = (\pi_1, \ldots, \pi_K)$, means $\mu_1, \ldots, \mu_K \in \mathbb{R}^d$ and covariance matrices $\Sigma_1, \ldots, \Sigma_K$
  - At time t we generate a new tree as follows:

$$c_t \sim \pi, \qquad x_t \sim N(\mu_{c_t}, \Sigma_{c_t})$$

Probabilistic Models

More generally:

- Θ consists of set of possible parameters
- We have a distribution P_θ over the data induced by each $\theta \in \Theta$
- Data is generated by one of the $\theta \in \Theta$
- Learning: Estimate value or distribution for $\theta^* \in \Theta$ given data

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Often referred to as frequentist view

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There are Bayesian and there Bayesians

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Maximize a posteriori probability of model given data

$$
\begin{aligned}
\theta_{MAP} &= \text{argmax}_{\theta \in \Theta} P(\theta | x_1, \ldots, x_n) \\
&= \text{argmax}_{\theta \in \Theta} \frac{P(x_1, \ldots, x_n | \theta) P(\theta)}{\sum_{\theta \in \Theta} P(x_1, \ldots, x_n | \theta) P(\theta)} \\
&= \text{argmax}_{\theta \in \Theta} \frac{P(x_1, \ldots, x_n | \theta) P(\theta)}{P(x_1, \ldots, x_n)} \\
&= \text{argmax}_{\theta \in \Theta} \underbrace{P(x_1, \ldots, x_n | \theta)}_{\text{likelihood}} \underbrace{P(\theta)}_{\text{prior}} \\
&= \text{argmax}_{\theta \in \Theta} \log P(x_1, \ldots, x_n | \theta) + \log P(\theta)
\end{aligned}
$$

Example: Gaussian Mixture Model

MLE: $\theta = (\mu_1, \ldots, \mu_K), \pi$

$$P_\theta(x_1, \ldots, x_n) = \prod_{t=1}^{n} \left(\sum_{i=1}^{K} \pi_i \frac{1}{\sqrt{(2 * 3.1415)^2 |\Sigma_i|}} \exp\left(-(x_t - \mu_i)^\top \Sigma_i (x_t - \mu_i)\right) \right)$$

EXAMPLE: GAUSSIAN MIXTURE MODEL

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MAP: with prior $\mu_i \sim N(0, \sigma I)$ and uniform prior on π

$$P(\theta | x_{1,\ldots,n}) = \prod_{t=1}^{n} \left(\sum_{i=1}^{K} \pi_i \frac{1}{\sqrt{(2 * 3.1415)^2 |\Sigma_i|}} \exp\left(-(x_t - \mu_i)^\top \Sigma_i (x_t - \mu_i)\right) \right)$$

$$\times \prod_{i=1}^{K} \frac{1}{\sqrt{(4 * 3.1415)^2}} \exp\left(-\|\mu_i\|^2\right)$$

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- In clustering for example, we can compute $P_{\theta^*}(c_t|x_t)$
- Hence we could assign to x_t cluster id c_t that has the largest probability. Inference step.
- *There are rough arguments*

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- We will come back to this later ...

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So why bother with the latent variables?

MLE for GMM

Let us consider the one dimensional case,

$$\log P_{\theta}(x_{1,\ldots,n}) = \sum_{t=1}^{n} \log \left(\sum_{i=1}^{K} \pi_i \frac{1}{\sqrt{(2 * 3.1415\sigma_i)^2}} \exp\left(-(x_t - \mu_i)^2/\sigma_i^2\right) \right)$$

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Now consider the partial derivative w.r.t. μ_1, we have:

$$\frac{\partial \log P_\theta(x_{1,\ldots,n})}{\partial \mu_1} = \sum_{t=1}^{n} \frac{\frac{\pi_1}{\sigma_1} \exp\left(-\frac{(x_t-\mu_1)^2}{\sigma_1^2}\right)}{\sum_{i=1}^{K} \frac{\pi_i}{\sigma_i} \exp\left(-\frac{(x_t-\mu_i)^2}{\sigma_i^2}\right)}$$

Even given all other parameters, optimizing w.r.t. just $\mu_!$ is hard!

MLE for GMM

Say by some magic you knew cluster assignments, then

$$\log P_\theta((x_t, c_t)_{1,\ldots,n}) = \sum_{t=1}^{n} \log\left(\frac{\pi_{c_t}}{\sqrt{(2 * 3.1415\sigma_{c_t})^2}} \exp\left(-\frac{(x_t - \mu_{c_t})^2}{2\sigma_{c_t}^2}\right)\right)$$

$$= \sum_{t=1}^{n} \left(\log(\pi_{c_t}) - \log(2 * 3.1415 * \sigma_{c_t}) - \frac{(x_t - \mu_{c_t})^2}{2\sigma_{c_t}^2}\right)$$

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Now consider the partial derivative w.r.t. μ_i, we have:

$$\frac{\partial \log P_\theta((x_t, c_t)_{1,\ldots,n})}{\partial \mu_i} = -\frac{\partial}{\partial \mu_i} \sum_{t=1}^{n}\left(\frac{1}{2\sigma_{c_t}^2}(x_t - \mu_{c_t})^2\right)$$

$$= -\frac{1}{2\sigma_i^2}\frac{\partial}{\partial \mu_i}\sum_{t:c_t=i}(x_t - \mu_i)^2$$

$$= \frac{1}{\sigma_1^2}\sum_{t:c_t=i}(x_t - \mu_i)\mu_i$$

MLE for GMM

- Optimize for σ_i and π, what do you get?

Towards EM Algorithm

- Say we are interested in either MLE or MAP estimators
- Latent variables can help, but we have a chicken and egg problem

Given all variables maximizing likelihood/a posteriori is easy

Given model parameter, optimizing distribution over the latent variables is easy