Machine Learning for Intelligent Systems

Lecture 6: Linear Classifiers and Perceptron

Reading: UML 9.1

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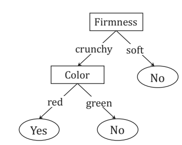
Hypothesis Spaces

(color = red) ∧ (size = small)

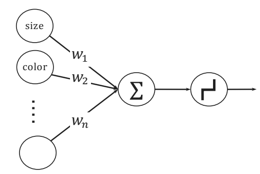
AND of feature-values

(color = red) ∨ (size = small)

OR of feature-values



Decision Trees



Linear Classifiers

Encoding in Euclidean Space

Represent apples as vectors in $\mathbb{R}^d$.

- Old:
  $X = \{A, B\} \times \{\text{red, green}\} \times \{\text{large, medium, small}\} \times \{\text{crunchy, soft}\}$.
- New:
  → $X \subseteq \mathbb{R}^4$.
  x_1 farm: $A \to 1$, $B \to -1$
  x_2 color: red $\to 1$, green $\to -1$
  x_3 size: large $\to 1$, medium $\to 0$, small $\to -1$
  x_4 firmness: crunchy $\to 1$, soft $\to -1$.
  → $Y = \{-1, +1\}$: Tasty $\to +1$, Not Tasty $\to -1$

Reuters Business News text classification:

- 9947 keywords (more accurately, word "stems")
- $X = \{0,1\}^{9947}$, where $x_i = 1$ if the keyword i appears in document.
- $Y = \{-1, +1\}$.

Linear Classifiers

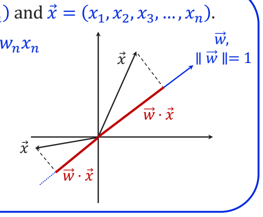
For a vector $\vec{w} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, the hypothesis $h_{\vec{w},b}: \mathbb{R}^d \to \mathbb{R}$ defined bellow is called a **linear classifier/linear predictor/halfspace,**

$$h_{\vec{w},b}(\vec{x}) = sign(\vec{w} \cdot \vec{x} + b) = \begin{cases} +1 & \vec{w} \cdot \vec{x} + b > 0 \\ -1 & \vec{w} \cdot \vec{x} + b \le 0 \end{cases}$$

Recall: Dot products

For two vectors: $\vec{w} = (w_1, w_2, w_3, \dots, w_n)$ and $\vec{x} = (x_1, x_2, x_3, \dots, x_n)$.

- $\vec{w} \cdot \vec{x} = w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n$
- $\vec{w} \cdot \vec{x}$ is the (signed) length of the projection of $\vec{x}$ on unit vector $\vec{w}$.



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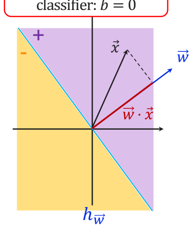
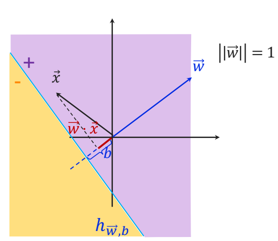
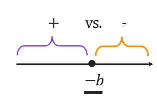
Homogenous linear classifier: $b = 0$



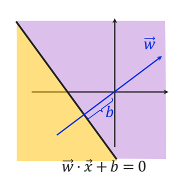


Linear Classifiers in all dimensions

- One-dimension: $h_{w,b}(x) = \text{sign}(wx + b)$
  → Decision boundary: point



- Two-dimension: $h_{\vec{w},b}(x) = \text{sign}(\vec{w} \cdot \vec{x} + b)$
  → Decision boundary: line



- d-dimension: $h_{\vec{w},b}(x) = \text{sign}(\vec{w} \cdot \vec{x} + b)$
  → Decision boundary: hyperplane $\vec{w} \cdot \vec{x} + b = 0$

Representational Power

Assume that x_1, x_2 take are binary values 0, 1. Represent the following using linear thresholds.

- $x_1 \wedge x_2$
- $x_1 \vee x_2$
- $x_1 \oplus x_2$

  → $\oplus$ represent XOR, where $x_1 \oplus x_2 = 1$ when exactly one of x_1 and x_2 is set to 1.

Homogenous vs. Non-homogenous

Any d-dimensional learning problem for **non-homogenous linear classifiers** has a **homogenous** form in (d+1) dimension.

Non-Homogenous $HS^d = \{h_{\vec{w},b} \mid \vec{w} \in \mathbb{R}^d, b \in \mathbb{R}\}$	**Homogenous** $HS^{d+1}_{homogenous} = \{h_{\vec{w}'} \mid \vec{w}' \in \mathbb{R}^{d+1}\}$
$\vec{x}$	$\vec{x}' = (\vec{x}, +1)$
$\vec{w}, b$	$\vec{w}' = (\vec{w}, b)$
$\vec{w} \cdot \vec{x} + b$	$\vec{w}' \cdot \vec{x}' = \vec{w} \cdot \vec{x} + b$

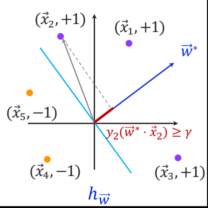
Without loss of generality, focus on **homogenous linear classifiers.**

Find a consistent classifier

If there is a homogeneous linear classifier that is consistent with $\{(\vec{x}_1, y_1), (\vec{x}_2, y_2), \ldots, (\vec{x}_m, y_m)\}$, how can we find it?

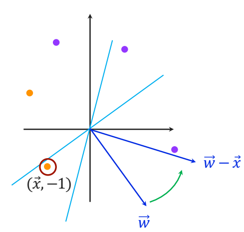
Unit vector $\vec{w}^*$ is such that for all $(\vec{x}_i, y_i)$, $\overbrace{y_i(\vec{w}^* \cdot \vec{x}_i) \geq \gamma > 0}^{\text{On the correct side, with wiggle room}}$.

We want to find a $\vec{w}$ such that $\underbrace{y_i(\vec{w} \cdot \vec{x}_i) > 0 \text{ for all } (\vec{x}_i, y_i).}_{\text{Can be done with a linear program}}$

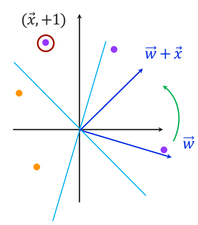


Improving a linear classifier

Start with a guess and improve it.



Move away from negative misclassified points



Move towards positive misclassified points

Frank Rosenblatt @ Cornell!

Perceptron (homogeneous & batch)

Input: Training data set $\{(\vec{x}_1, y_1), (\vec{x}_2, y_2), \ldots, (\vec{x}_m, y_m)\}$
Initialize $\vec{w}^{(0)} = (0, \ldots, 0),\ t = 0$
While there is $i \in [m]$, such that $\underbrace{y_i(\vec{w}^{(t)} \cdot \vec{x}_i) \leq 0}_{\text{misclassified}}$ then,

- $\vec{w}^{(t+1)} = \vec{w}^{(t)} + y_i \vec{x}_i \begin{cases} \vec{w}^{(t)} + \vec{x}_i & \text{for positive instances} \\ \vec{w}^{(t)} - \vec{x}_i & \text{for negative instances} \end{cases}$
- $t \leftarrow t + 1$

End While
Output $\vec{w}^{(t)}$

Example: Reuters Text Classification