

# Support Vector Machines: Kernels

CS4780/5780 – Machine Learning  
Fall 2012

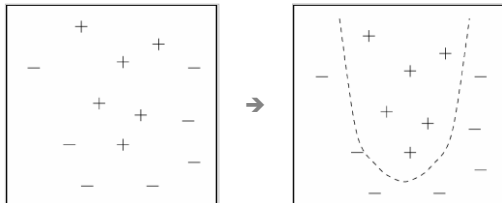
Thorsten Joachims  
Cornell University

Reading: Schoelkopf/Smola Chapter 7.4, 7.6, 7.8  
Cristianini/Shawe-Taylor 3.1, 3.2, 3.3.2, 3.4

## Outline

- Transform a linear learner into a non-linear learner
- Kernels can make high-dimensional spaces tractable
- Kernels can make non-vectorial data tractable

## Non-Linear Problems

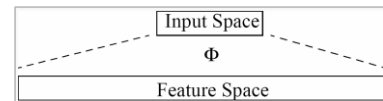


Problem:

- some tasks have non-linear structure
  - no hyperplane is sufficiently accurate
- How can SVMs learn non-linear classification rules?

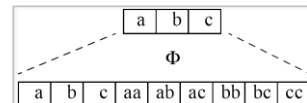
## Extending the Hypothesis Space

Idea: add more features



→ Learn linear rule in feature space.

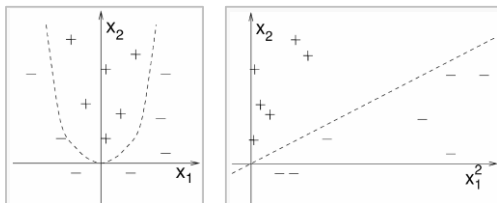
Example:



→ The separating hyperplane in feature space is degree two polynomial in input space.

## Example

- Input Space:  $\vec{x} = (x_1, x_2)$  (2 attributes)
- Feature Space:  $\Phi(\vec{x}) = (x_1^2, x_2^2, x_1, x_2, x_1 x_2, 1)$  (6 attributes)



## Dual SVM Optimization Problem

- Primal Optimization Problem

$$\begin{aligned} \text{minimize: } & P(\vec{w}, b, \xi) = \frac{1}{2} \vec{w} \cdot \vec{w} + C \sum_{i=1}^n \xi_i \\ \text{subject to: } & \forall_{i=1}^n : y_i(\vec{w} \cdot \vec{x}_i + b) \geq 1 - \xi_i \\ & \forall_{i=1}^n : \xi_i > 0 \end{aligned}$$

- Dual Optimization Problem

$$\begin{aligned} \text{maximize: } & D(\vec{\alpha}) = \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n y_i y_j \alpha_i \alpha_j (\vec{x}_i \cdot \vec{x}_j) \\ \text{subject to: } & \sum_{i=1}^n y_i \alpha_i = 0 \\ & \forall_{i=1}^n : 0 \leq \alpha_i \leq C \end{aligned}$$

- Theorem: If  $\vec{w}^*$  is the solution of the Primal and  $\vec{\alpha}^*$  is the solution of the Dual, then

$$\vec{w}^* = \sum_{i=1}^n \alpha_i^* y_i \vec{x}_i$$

## Kernels

- Problem:
    - Very many Parameters! Polynomials of degree  $p$  over  $N$  attributes in input space lead to  $O(Np)$  attributes in feature space!
  - Solution:
    - The dual OP depends only on inner products
    - Kernel Functions  $K(\vec{a}, \vec{b}) = \Phi(\vec{a}) \cdot \Phi(\vec{b})$
  - Example:
    - For  $\Phi(\vec{x}) = (x_1^2, x_2^2, \sqrt{2}x_1, \sqrt{2}x_2, 1)$  calculating  $K(\vec{a}, \vec{b}) = [\vec{a} \cdot \vec{b} + 1]^2$  computes inner product in feature space.
- no need to represent feature space explicitly.

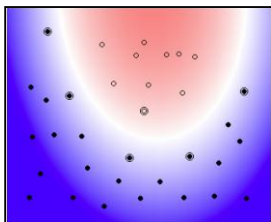
## SVM with Kernel

- Training:
 
$$\begin{aligned} \text{maximize: } D(\vec{\alpha}) &= \sum_{i=1}^n \alpha_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n y_i y_j \alpha_i \alpha_j K(\vec{x}_i, \vec{x}_j) \\ \text{subject to: } & \sum_{i=1}^n y_i \alpha_i = 0 \\ & \forall_{i=1}^n : 0 \leq \alpha_i \leq C \end{aligned}$$
- Classification:
 
$$\begin{aligned} h(\vec{x}) &= \text{sign} \left( \sum_{i=1}^n \alpha_i y_i \Phi(\vec{x}_i) \cdot \Phi(\vec{x}) + b \right) \\ &= \text{sign} \left( \sum_{i=1}^n \alpha_i y_i K(\vec{x}_i, \vec{x}) + b \right) \end{aligned}$$
- New hypotheses spaces through new Kernels:
  - Linear:  $K(\vec{a}, \vec{b}) = \vec{a} \cdot \vec{b}$
  - Polynomial:  $K(\vec{a}, \vec{b}) = [\vec{a} \cdot \vec{b} + 1]^d$
  - Radial Basis Function:  $K(\vec{a}, \vec{b}) = \exp(-\gamma[\vec{a} - \vec{b}]^2)$
  - Sigmoid:  $K(\vec{a}, \vec{b}) = \tanh(\gamma[\vec{a} \cdot \vec{b}] + c)$

## Examples of Kernels

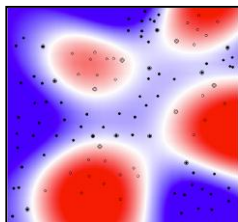
Polynomial

$$K(\vec{a}, \vec{b}) = [\vec{a} \cdot \vec{b} + 1]^2$$



Radial Basis Function

$$K(\vec{a}, \vec{b}) = \exp(-\gamma[\vec{a} - \vec{b}]^2)$$



## What is a Valid Kernel?

Definition: Let  $X$  be a nonempty set. A function is a valid kernel in  $X$  if for all  $n$  and all  $x_1, \dots, x_n \in X$  it produces a Gram matrix

$$G_{ij} = K(x_i, x_j)$$

that is symmetric

$$G = G^T$$

and positive semi-definite

$$\forall \vec{\alpha}: \vec{\alpha}^T G \vec{\alpha} \geq 0$$

## How to Construct Valid Kernels

Theorem: Let  $K_1$  and  $K_2$  be valid Kernels over  $X \times X$ ,  $\alpha \geq 0$ ,  $0 \leq \lambda \leq 1$ ,  $f$  a real-valued function on  $X$ ,  $\phi: X \rightarrow \mathbb{R}^m$  with a kernel  $K_3$  over  $\mathbb{R}^m \times \mathbb{R}^m$ , and  $K$  a symmetric positive semi-definite matrix. Then the following functions are valid Kernels

$$K(x, z) = \lambda K_1(x, z) + (1-\lambda) K_2(x, z)$$

$$K(x, z) = \alpha K_1(x, z)$$

$$K(x, z) = K_1(x, z) K_2(x, z)$$

$$K(x, z) = f(x) f(z)$$

$$K(x, z) = K_3(\phi(x), \phi(z))$$

$$K(x, z) = x^T K z$$

## Kernels for Discrete and Structured Data

Kernels for Sequences: Two sequences are similar, if the have many common and consecutive subsequences.

Example [Lodhi et al., 2000]: For  $0 \leq \lambda \leq 1$  consider the following features space

	c-a	c-t	a-t	b-a	b-t	c-r	a-r	b-r
$\phi(\text{cat})$	$\lambda^2$	$\lambda^3$	$\lambda^2$	0	0	0	0	0
$\phi(\text{car})$	$\lambda^2$	0	0	0	0	$\lambda^3$	$\lambda^2$	0
$\phi(\text{bat})$	0	0	$\lambda^2$	$\lambda^2$	$\lambda^3$	0	0	0
$\phi(\text{bar})$	0	0	0	$\lambda^2$	0	0	$\lambda^2$	$\lambda^3$

⇒  $K(\text{car}, \text{cat}) = \lambda^4$ , efficient computation via dynamic programming

## Kernels for Non-Vectorial Data

- Applications with Non-Vectorial Input Data
    - classify non-vectorial objects
    - Protein classification (x is string of amino acids)
    - Drug activity prediction (x is molecule structure)
    - Information extraction (x is sentence of words)
    - Etc.
  - Applications with Non-Vectorial Output Data
    - predict non-vectorial objects
    - Natural Language Parsing (y is parse tree)
    - Noun-Phrase Co-reference Resolution (y is clustering)
    - Search engines (y is ranking)
- Kernels can compute inner products efficiently!

## Properties of SVMs with Kernels

- Expressiveness
  - SVMs with Kernel can represent any boolean function (for appropriate choice of kernel)
  - SVMs with Kernel can represent any sufficiently “smooth” function to arbitrary accuracy (for appropriate choice of kernel)
- Computational
  - Objective function has no local optima (only one global)
  - Independent of dimensionality of feature space
- Design decisions
  - Kernel type and parameters
  - Value of C

## SVMs for other Problems

- Multi-class Classification
  - [Schoelkopf/Smola Book, Section 7.6]
- Regression
  - [Schoelkopf/Smola Book, Section 1.6]
- Outlier Detection
  - D.M.J. Tax and R.P.W. Duin, "Support vector domain description", Pattern Recognition Letters, vol. 20, pp. 1191-1199, 1999b. 26
- Structured Output Prediction
  - B. Taskar, C. Guestrin, D. Koller - Advances in Neural Information Processing Systems, 2003.
  - I. Tsochantaris, T. Hofmann, T. Joachims, and Y. Altun, Support Vector Machine Learning for Interdependent and Structured Output Spaces, Proceedings of the International Conference on Machine Learning (ICML), 2004.