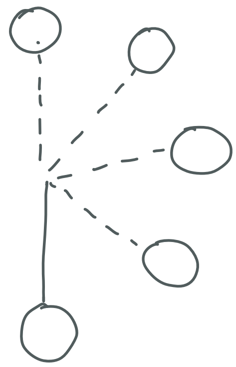
$t = T$



$t = 0$


STATE $\quad x_t = \begin{bmatrix} \theta \\ \dot\theta \end{bmatrix}$

ACTIONS: $u_t : \tau$

torque.


DYNAMICS

$$mgl\sin\theta + \tau = I\ddot\theta$$

$$mgl\sin\theta + \tau = ml^2\ddot\theta$$

$$\boxed{\ddot\theta = \frac{g}{l}\sin\theta + \frac{\tau}{ml^2}} \approx \frac{g}{l}\theta + \frac{\tau}{ml^2}$$

$$x_{t+1} = f(x_t, u_t) \longrightarrow \dot{x}_t = f(x_t, u_t)$$

$$\begin{bmatrix} \theta \\ \dot\theta \end{bmatrix}$$

$$\langle S, A, \tau, C \rangle$$

integrate this 2 times

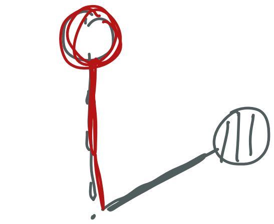
$$\ddot\theta = \frac{g}{l}\theta + \frac{\tau}{ml^2}$$

$$x_{t+1} = f(x_t, u_t)$$

$$\begin{bmatrix} \theta \\ \dot\theta \end{bmatrix}_{t+1} = \underbrace{\begin{bmatrix} \left(1+\frac{1}{2}g\Delta t^2\right) & \Delta t \\ \frac{g}{l}\Delta t & 1 \end{bmatrix}_{2\times 2}}_{A} \begin{bmatrix} \theta \\ \dot\theta \end{bmatrix}_t + \underbrace{\frac{1}{ml^2}\begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix}_{2\times 1}}_{B} [\tau]_t$$

$$x_{t+1} = Ax_t + Bu_t \quad \text{(LINEAR)!}$$

## Cost



$$\underline{\theta^2 + \dot{\theta}^2} + r\tau^2$$

$$\underbrace{\begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}}_{x_t^T} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}}_{x_t} + \overset{[\tau]}{u^T} \overset{(r)}{R} \overset{[\tau]}{u}$$

$$c(x_t, u_t) = x_t^T \underline{Q} x_t + u_t^T \underline{R} u_t \quad \text{(QUADRATIC)}$$

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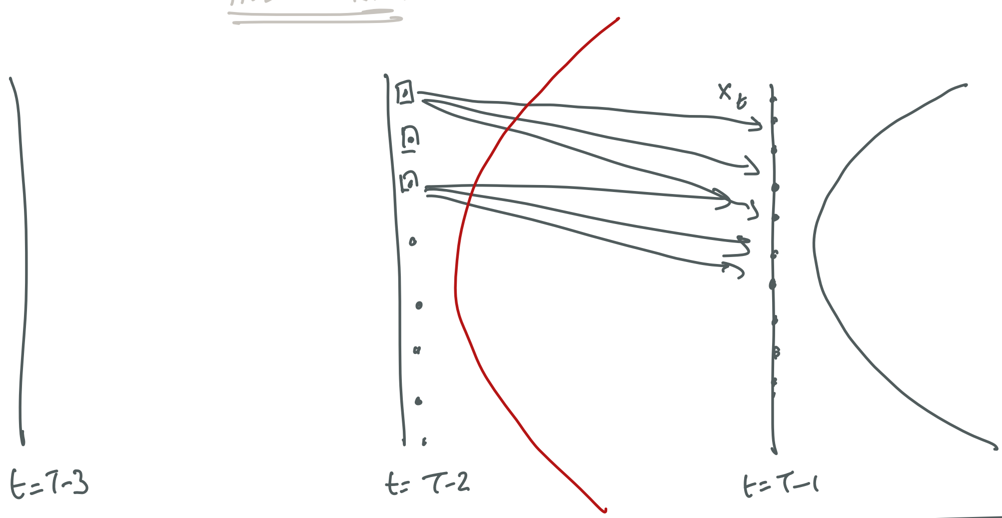
## The Trick

$V^*(x_{T-1})$ is a Quadratic

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## Proof

(1) Show that the value function is Quadratic at $T-1$

(2) Show that if value function is Quad at $t+1$ then it is Quadratic at $t$.

$$V_{T-1}(x_{T-1}) = \min_{u_{T-1}} \left[ c(x_{T-1}, u_{T-1}) + 0 \right]$$

$$= \min_{u_{T-1}} \left[ \underline{x_{T-1}^T Q x_{T-1}} + u_{T-1}^T R u_{T-1} \right]$$

$$\frac{\partial}{\partial u_{T-1}} \left( 2 R u_{T-1} \right) = 0 \Rightarrow \boxed{u_{T-1} = 0}$$

$$V_{T-1}(x_{T-1}) = \underline{x_{T-1}^T Q x_{T-1}} + 0$$

$$q\, x_{t+1}^2$$

---

(2) __At timestep t__

$$V_t(x_t) = \min_{u_t} \left[ c(x_t, u_t) + \underline{V_{t+1}}(x_{t+1}) \right]$$

$$= \min_{u_t} \left[ \underline{x_t^T Q x_t} + u_t^T R u_t + \underline{x_{t+1}^T V_{t+1} x_{t+1}} \right]$$

QUADRATIC QUADRATIC QUADRATIC

$$\frac{\partial}{\partial u_t} (\cdot) = 0$$

$$x_{t+1} = A x_t + B u_t$$

$$\cancel{2} u_t^T R + \cancel{2} (x_{t+1}^T) V_{t+1} B = 0$$

$$R u_t + B^T V_{t+1} (A x_t + B u_t) = 0$$

$$(R + B^T V_{t+1} B) u_t = - B^T V_{t+1} A x_t$$

$$u_t = \underbrace{\boxed{-(R + B^T V_{t+1} B)^{-1} B^T V_{t+1} A}} x_t$$

$$u_t = K_t x_t$$

(LINEAR)

$$\min_{u_t} \left[ \underline{\underline{x_t^T Q x_t}} + u_t^T R u_t + \underline{\underline{x_{t+1}^T V_{t+1} x_{t+1}}} \right]$$

$$V_t(x_t) = x_t^T ( - \; - \; - \; - \; - \; - \; - ) x_t$$