These notes are mainly based on [2], [3], [4] and [1].
So far we have studied robotic mechanisms from a purely kinematic perspective. We presented methods (forward/inverse kinematics) for deriving equations relating the control input (joint angles or joint angle velocities) with the output motion of a robotic mechanism (cartesian positions or velocities). This kinematic analysis did not incorporate the effect of external loads on the state of a robot. However, robots operating in real-world environments, are constantly subject to forces and torques, resulting from the robot’s own components or from the robot’s interactions with its environment. Therefore, in order to derive a realistic model of a robot and design control algorithms for performing useful tasks, it is necessary to study a robot from a dynamics perspective.

In this chapter, we present methods for deriving models of robot dynamics. These models are sets of equations, commonly referred to as equations of motion, that relate the loads applied on a mechanism to its kinematic state.

1 Statics
Static is a branch of mechanics that studies systems in static equilibrium. Many tasks in robotics can be considered to be static or quasi-static\(^1\). In practice, this allows us to derive the torques that are necessary to exert a desired force to the environment or study how forces are transmitted in a robotic mechanism.

1.1 Equivalent Joint Torques
Consider the n-DoF (degree of freedom) manipulator depicted in Fig. 1a. Assume no friction at the joints and no gravity. The manipulator is in static equilibrium, i.e., the torques on its joints \( \tau = [\tau_1, ..., \tau_n]^T \in \mathbb{R}^n \) result to a force \( f \in \mathbb{R}^3 \) and a moment \( \mu \in \mathbb{R}^3 \), exerted by the end effector to the environment (see Fig. 1b). From Newton’s third law, the environment exerts to the end effector equal force and moment of the same magnitude, towards the opposite directions. This force-moment pair can be combined into a vector \( F = [f^T \mu^T]^T \in \mathbb{R}^6 \) that is called wrench.

It can be shown that the joint torques are related to the wrench exerted by the end-effector as follows:

\[
\tau = J^T \cdot F
\]

where \( J \) denotes the \( 6 \times n \) Jacobian matrix relating infinitesimal joint displacements \( dq \) to infinitesimal end effector displacements \( dp \), i.e.,

\[
dp = Jdq.
\]

\(^1\)Loadings in which inertial effects can be assumed to be negligible.
Eq. (1) can be proven by employing the principle of virtual work. Consider virtual joint displacements \( \delta q = [\delta q_1, ..., \delta q_n]^T \) and virtual displacements\(^2\) at the end-effector (translational and angular, see Fig. 2) \( \delta p = [\delta x_e^T, \delta \phi_e^T]^T \). Assume that the joints and the end-effector are only moved in the directions that are geometrically admissible. Then the virtual work produced by the moments, forces and torques can be written as:

\[
\delta W = \tau_1 \cdot \delta q_1 + \tau_2 \cdot \delta q_2 + ... + \tau_n \cdot \delta q_n - f^T \cdot \delta x_e - \mu^T \cdot \delta \phi_e = \tau^T \cdot \delta q - F^T \cdot \delta p \quad (3)
\]

Substituting eq. (2) into eq. (3), we get:

\[
\delta W = \tau^T \delta q - F^T J \cdot \delta q = (\tau - J^T F)^T \cdot \delta q \quad (4)
\]

According to the principle of virtual work, the system is in equilibrium if and only if the virtual work \( \delta W \) is zero for arbitrary virtual displacements that obey the geometric constraints imposed by the structure of the mechanism. This is always true if and only if eq. (1) is true.

\(^2\)Virtual displacements, denoted here with \( \delta \), to differentiate them from actual displacements (denoted with \( d \)) must only satisfy geometric constraints; they do not have to comply with laws of motion.
1.2 Force Transmission

The static equilibrium of a robotic system implies that all of its components are also in static equilibrium. Therefore, each link of a manipulator in equilibrium must also individually be in a state of equilibrium. From the force balance, we get:

$$f_i - f_{i+1} + m_i g = 0,$$

where $g$ is gravity and all force vectors are expressed with respect to a frame attached at axis $i$. Eq. (5) can then be rewritten as:

$$f_i^i = R_{i+1}^i f_{i+1}^i + m_i g^i.$$

The balance of moments around the centroid $C_i$ yields:

$$\mu_i - \mu_{i+1} - p_{C_i}^i \times f_i + (p_{i+1}^i - p_{C_i}^i) \times (-f_{i+1}) = 0$$

(7)

Eq. (7) can be rewritten as:

$$\mu_i^i = R_{i+1}^i \mu_{i+1} + S(p_{C_i}^i) f_i^i - S(p_{i+1}^i - p_{C_i}^i) R_{i+1}^i f_{i+1}^i.$$

(8)

where

$$S(r) = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

(9)
Figure 3: 

is the skew-symmetric matrix of a vector \( r \in \mathbb{R}^3 \). Substituting eq. (6) into eq. (8), we get:

\[
\mu_i = R_{i+1}^i \mu_{i+1}^{i+1} + S(p_{C_i}^i)(R_{i+1}^i f_{i+1}^{i+1} + m_i g_i) - S(p_{i+1}^i - p_{C_i}^i)R_{i+1}^i f_{i+1}^{i+1}
\]  

(10)

Eq. (10) can be further simplified to yield:

\[
\mu_i = R_{i+1}^i \mu_{i+1}^{i+1} - m_i S(p_{C_i}^i) g_i + S(p_{i+1}^i)R_{i+1}^i f_{i+1}^{i+1}
\]  

(11)

Neglecting gravity, eqs. (5) and (11) can be written as:

\[
\begin{bmatrix}
  f_i^i \\
  \mu_i^i
\end{bmatrix} = 
\begin{bmatrix}
  R_{i+1}^i & 0 \\
  S(p_{i+1}^i)R_{i+1}^i & R_{i+1}^i
\end{bmatrix} \begin{bmatrix}
  f_{i+1}^{i+1} \\
  \mu_{i+1}^{i+1}
\end{bmatrix}.
\]  

(12)

This equation connects the forces and moments applied at a link for a manipulator in equilibrium.

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3The skew-symmetric matrix allows us to transform a cross product of two vectors into a dot product of a matrix with a vector, i.e., for \( a, b \in \mathbb{R}^3 \), it holds that \( a \times b = S(a)b \).
2 Dynamics

2.1 Kinetic and Potential Energy

2.1.1 Kinetic Energy and Potential Energy of a Rigid Body

In general, the motion of a rigid body in the 3-dimensional cartesian space is the result of a translational velocity $v$ and an angular velocity $\omega$, usually defined about its center of mass (see Fig. 4). These velocities correspond to a translational and a rotational kinetic energy respectively. Their sum corresponds to the total kinetic energy of the body:

$$K = \frac{1}{2}mv^T v + \frac{1}{2}\omega^T I\omega$$  \hspace{1cm} (13)

where $m$ is the body’s mass and $I$ is the body’s inertia tensor, defined with respect to a body frame attached at its center of mass. All velocities along with the inertia tensor are expressed with respect to the global inertial frame.

Furthermore, a rigid body possesses potential energy, as a result of its position with respect to some assumed datum. For the body in Fig. 4, the expression for the potential energy is the following:

$$P = mg^T r_C.$$  \hspace{1cm} (14)

2.1.2 The Inertia Tensor

The inertia tensor is a symmetric, $3 \times 3$ matrix, essentially quantifying the resistance of the body to any change of its orientation. An inertia tensor is usually given with respect to the body frame, therefore we often need to express it with respect to an appropriate global frame. This can be done with a similarity transformation as follows:

$$I = RIR^T$$  \hspace{1cm} (15)

where $I$ is the inertia tensor expressed with respect to the body frame and $R$ is the rotation matrix defining the body’s orientation with respect to the global frame.

The inertia tensor $I$ can be derived as:

$$I = \begin{bmatrix}
I_{xx} & I_{xy} & I_{xz} \\
I_{yx} & I_{yy} & I_{yz} \\
I_{zx} & I_{zy} & I_{zz}
\end{bmatrix}$$  \hspace{1cm} (16)
Figure 4: A rigid body moving with linear velocity $v$ and angular velocity $\omega$.

with

$$I_{xx} = \int \int \int (y^2 + z^2)\rho(x, y, z) dx \, dy \, dz,$$

(17)

$$I_{yy} = \int \int \int (x^2 + z^2)\rho(x, y, z) dx \, dy \, dz,$$

(18)

$$I_{zz} = \int \int \int (x^2 + y^2)\rho(x, y, z) dx \, dy \, dz$$

(19)

and

$$I_{xy} = I_{yx} = \int \int \int xy\rho(x, y, z) dx \, dy \, dz,$$

(20)

$$I_{xz} = I_{zx} = \int \int \int xz\rho(x, y, z) dx \, dy \, dz,$$

(21)

$$I_{yz} = I_{zy} = \int \int \int yz\rho(x, y, z) dx \, dy \, dz,$$

(22)

where $\rho(x, y, z)$ is the mass density function representing the distribution of mass along the body.

For typical types of objects such as spheres, cubes, rods etc, the inertia tensor can be found from tables. For objects synthesized from such shapes, the inertia tensor can be computed through the use of appropriate theorems (e.g. the parallel axis theorem).
2.1.3 Kinetic Energy and Potential Energy of a n-link Manipulator

Using the previous definitions, we may derive expressions for the Kinetic and Potential energy of a manipulator. Assuming that $J_{v_i}$ and $J_{\omega_i}$ are appropriate Jacobian matrices relating the joint velocities with the linear ($v_i$) and angular velocity ($\omega_i$) of link $i$:

$$v_i = J_{v_i}(q)\dot{q}, \quad \omega_i = J_{\omega_i}(q)\dot{q}, \quad (23)$$

the expression for the kinetic energy of the whole n-link manipulator can be derived as follows:

$$K = \frac{1}{2} \dot{q}^T \left[ \sum_{i=1}^{n} \left\{ m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega_i}(q) \right\} \right] \dot{q} \quad (24)$$

$$= \dot{q}^T D(q) \dot{q} \quad (25)$$

where $m_i$ and $I_i R_i$ are the mass, inertia tensor and orientation (rotation matrix representing the orientation of frame $i$ with respect to the global frame) of link $i$ and

$$D(q) = \sum_{i=1}^{n} \left\{ m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega_i}(q) \right\}. \quad (26)$$

The matrix $D(q)$, called the inertia matrix, is a symmetric, positive definite, $n \times n$ matrix that depends on the configuration of the robot.

The potential energy of a manipulator can be derived using eq. (14) as follows:

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} m_i g^T r_{C_i} \quad (27)$$

where $P_i$ is the potential energy of link $i$ and $r_{C_i}$ is the position of its center of mass with respect to the global frame.

2.2 The Euler-Lagrange Equations

The Euler-Lagrange equations is are a set of second-order partial differential equations that are widely used in mechanics to derive the equations of motion of a system. Their general formulation requires the description of a system with respect to a set of generalized coordinates $(q_1, ..., q_n)$, where $n$ is the number of degrees of freedom of the system and the definition of a functional $\mathcal{L} = K - P$, called the Lagrangian, with $K$ and $P$ being the expressions for the kinetic and the potential energy of the system. Then the Euler-Lagrange equations are defined as:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = \tau_k, \quad k = 1, 2, ..., n, \quad (28)$$

where $\tau_k$ denotes a generalized force associated with the $q_k$ coordinate.
2.3 Equations of Motion

The Euler-Lagrange equations provide a useful tool that allows us to derive the equations of motion of robotic systems. They are not the only way to do so; other alternatives include Newton’s second law, the Newton-Euler method and the Principle of Virtual Work. The applicability and the advantages of each depend on the specifics of the system and are related to issues such as computational efficiency. In this course we limit our study to the Euler-Lagrange equations as they probably constitute the most widely used tool to derive the equations of motion in robotics. In this section, we demonstrate their use with a simple example and derive general expressions of the equations of motion for an n-link manipulator using them.

2.3.1 Example: 1 DoF system

The ball depicted in Fig. 5 is subject to its weight \( mg \) and an external force \( f \). Notice that in such a simple case, the system’s equations of motion can be quickly derived from Newton’s second law as:

\[
m\ddot{y} = f - mg. \tag{29}
\]

We will validate this result using the Euler-Lagrange equations.

It is clear that the behavior of the system can be described with just one generalized coordinate, its height \( y \). So, we derive expressions for the system’s kinetic and potential energy and of course for its Lagrangian with respect to \( y \):

\[
K = \frac{1}{2} m\dot{y}^2, \quad P = mgy, \quad \mathcal{L} = \frac{1}{2} m\dot{y}^2 - mgy. \tag{30}
\]

The next step is to derive the partial derivatives:

\[
\frac{\partial \mathcal{L}}{\partial \dot{y}} = m\dot{y}, \quad \frac{\partial \mathcal{L}}{\partial y} = -mg, \tag{31}
\]

using which we can write the Euler-Lagrange equation of the system as:

\[
\frac{d}{dt} (m\dot{y}) + mg = f. \tag{32}
\]

Calculating the time derivative and reordering we get:

\[
m\ddot{y} = f - mg, \tag{33}
\]

which is the same as the expression derived with Newton’s law.
2.3.2 Dynamic Model of a n-link Manipulator

The expression for the kinetic energy of a n-link manipulator was derived in eq. (25). That expression can be rewritten as:

\[ K = \frac{1}{2} \dot{q}^T D(q) \dot{q} = \sum_{i,j} d_{ij}(q) \dot{q}_i \dot{q}_j \]  

(34)

where \( d_{ij} \) are the elements of the inertia matrix \( D(q) \). We also derived an expression for the potential energy as a function of the configuration \( q \), i.e., \( P = P(q) \). The Lagrangian can then be derived as:

\[ \mathcal{L} = K - P = \sum_{i,j} d_{ij}(q) \dot{q}_i \dot{q}_j - P(q). \]  

(35)

We derive the partial derivative of the Lagrangian with respect to the \( k \)th joint velocity as:

\[ \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_j d_{kj} \dot{q}_j, \quad j = 1, \ldots, n \]  

(36)

and using the product rule and then the chain rule, we derive the time derivative of this as:

\[ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \sum_j d_{kj} \ddot{q}_j + \sum_j \frac{d}{dt} d_{kj} \dot{q}_j \]  

(37)

\[ = \sum_j d_{kj} \ddot{q}_j + \sum_j \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j \]  

(38)
In a similar fashion, the partial derivative of the Lagrangian with respect to the $k$th joint position is found as:

$$\frac{\partial L}{\partial q_k} = \frac{1}{2} \sum_{i,j} d_{ij} \dot{q}_i \dot{q}_j - \frac{\partial P}{\partial q_k}. \quad (39)$$

Now we can write the Lagrange equations as:

$$\sum_j d_{kj} \ddot{q}_j + \sum_{i,j} \left\{ \frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2} \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k, \ k = 1, ..., n \quad (40)$$

Upon a few algebraic manipulations, the previous expression can be transformed to:

$$\sum_j d_{kj} \ddot{q}_j + \sum_{i,j} c_{ijk}(q) \dot{q}_i \dot{q}_j + g_k = \tau_k, \ k = 1, ..., n \quad (41)$$

where

$$g_k = \frac{\partial P}{\partial q_k} \quad (42)$$

and

$$c_{ijk} = \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \quad (43)$$

The symbols $c_{ijk}$ are known as Christoffel symbols of the first kind. One notable property of these is that $c_{ijk} = c_{jik}$. This equation is composed of three different types of terms: (1) terms involving the second derivatives of $q$, (2) terms involving quadratic terms of the first derivatives of $q$ with coefficients that may depend on $q$ and (3) terms involving only $q$ and not its derivatives. Regarding the quadratic terms (type 2), they can be further classified into (a) those involving a product of the form $\dot{q}_i^2$--called centrifugal terms-- and (b) those involving a product of the form $\dot{q}_i \dot{q}_j (i \neq j)$,--called coriolis terms.

The equations are commonly written in the following form:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \quad (44)$$

where the $(k,j)$ element of the matrix $C(q, \dot{q})$ is given by:

$$c_{jk} = \sum_{i=1}^n c_{ijk}(q) \dot{q}_i = \sum_{i=1}^n \frac{1}{2} \left\{ \frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right\} \dot{q}_i \quad (45)$$

and the gravity term is defined as:

$$g(k) = [g_1(q), ..., g_n(q)]^T. \quad (46)$$
2.4 Long-Form Examples

Example 1. Seesaw

Concepts Reviewed: moments, moment of inertia, Euler-Lagrange Equations, conservation of energy

![Seesaw Diagram](image)

Figure 6: Seesaw used in Example 1

Problem: Consider a seesaw with two masses of $m_1 = 20$ kg and $m_2 = 40$ kg resting on either side, as in Fig. 6. The mass $m_1$ is resting on the ground, which is $\frac{1}{2}$ m below the fulcrum. Assume the mass of the seesaw is negligible. Answer the following questions.

1. At the configuration shown in Fig. 6, with the seesaw initially at rest, what will be the net wrench on the seesaw due to gravity?

2. Suppose there are no losses due to friction or wind resistance. What will be the velocity of the seesaw when it reaches a horizontal configuration?

3. Suppose the heavier mass falls off at the instant the seesaw is horizontal. What is the acceleration of the seesaw immediately following?

4. At what angle will the seesaw reach zero velocity before it starts to reverse direction.

Solution: We address the questions in turn.

1. We choose to place the reference point at the fulcrum, about which the seesaw rotates.

   At time $t = 0$, the seesaw has not yet begun to move, so this can be addressed as a statics problem. The forces on the system are two downward vectors through the two masses due to gravity, and an upward force through
the fulcrum in the form of a constraint against translation at the pivot point. The constraint will precisely negate the (linear) force due to gravity, leaving a pure torque on the seesaw. The free-body diagram is shown below.

In this configuration, the net moment about the fulcrum is

\[ \mu_f = r_1 \times f_1 + r_2 \times f_2 \]

\[ \begin{bmatrix} -\sqrt{1-h^2} & 0 & 0 & 0 & 0 \\ -h & -m_1g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -m_1g \\ 0 \\ h \\ 0 \end{bmatrix} \]

\[ = \begin{bmatrix} -\frac{\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ -0.5 & -20g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\sqrt{3}}{2} \\ 0.5 \end{bmatrix} \times \begin{bmatrix} 0 \\ -40g \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

\[ = \begin{bmatrix} 0 \\ 0 \\ -10\sqrt{3}g \end{bmatrix} \]

The overall wrench is \( w = [f_f^T \quad \mu_f^T]^T = [0 \quad 0 \quad 0 \quad 0 \quad -10\sqrt{3}g]^T \).

Using Euler-Lagrange. We can also examine this problem using the Euler-Lagrange equations of motion. First, we must establish a set of generalized coordinates that eliminate all constraints. There is only one degree of freedom in this system, so the configuration in generalized coordinates is \( q = [\theta] \), namely the angle of the seesaw. We can write the Lagrangian by computing kinetic and potential energies as

\[ \mathcal{L} = \mathcal{K} - \mathcal{P} \]

\[ \mathcal{K} = \frac{1}{2}mv^T v + \frac{1}{2}I\omega^2 \]

\[ \mathcal{P} = \sum m_i g r_i \]
Although the kinetic energy is zero in this static example, we compute it because it will be needed later anyway. The linear velocity term \( v = 0 \) due to the fulcrum constraint. The angular velocity is simply \( \omega = \dot{\theta} \). The moment of inertia \( I \) requires an integral over the 2D area of the seesaw. Since we have only two point-masses, we can simplify the computation using a summation, as

\[
I = \sum i m_i r_i^T r_i = m_1 + m_2
\]  

(54)

Thus,

\[
K = \frac{1}{2} (m_1 + m_2) \dot{\theta}^2 
\]  

(55)

\[
P = -m_1 g \sin \theta + m_2 g \sin \theta = (m_2 - m_1) g \sin \theta
\]  

(56)

Note that the potential energy was computed by assuming that the horizontal plane passing from the fulcrum corresponds to zero potential energy\(^4\) and that we simplified the equation by removing the length of the lever arms, each of which is 1 m.

Now, in order to derive the torque acting on the system when at rest at this configuration, we need to make the assumption of an external torque \( \tau \), with-standing the natural tendency of the system to rotate clockwise. With this assumption, the equation of motion of the system can be written as:

\[
\tau = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} 
\]  

(57)

\[
\tau = \frac{d}{dt} \left( \frac{1}{2} (m_1 + m_2) \dot{\theta}^2 - (m_2 - m_1) g \sin \theta \right) 
\]  

\[- \frac{\partial}{\partial \theta} \left( \frac{1}{2} (m_1 + m_2) \dot{\theta}^2 - (m_2 - m_1) g \sin \theta \right) 
\]  

(58)

\[
= (m_2 - m_1) g \cos \theta. 
\]  

(59)

Note that the expression we derived for \( \tau \) corresponds to the torque required to withstand the roque exerted to the seesaw by the system of point masses. Therefore, the torque exerted by the masses will be the exact opposite quantity, i.e., \(-(m_2 - m_1) g \cos \theta\). Substituting \( \theta = \sin^{-1} \frac{h}{l} = 30^\circ \), we can see again that \( \tau = -20 g \cos(30) = -10\sqrt{3} g \text{ N-m} \). This is a clockwise torque.

\(^4\)This is an assumption; any plane could serve as a datum of zero potential energy as long as we are consistent in all our calculations.
2. Since the torque on the seesaw is clockwise, the right side will start to fall. To find the velocity when it reaches a horizontal configuration, it is simply a physics problem. An algorithmic implementation could numerically integrate the equation of motion obtained from the Euler-Lagrange equation. This would be the best approach if time is involved.

We can instead compute the total energy as

\[
\mathcal{K}(q, \dot{q}) + \mathcal{P}(q) = \frac{1}{2} (m_1 + m_2) \dot{\theta}^2 + (m_2 - m_1) g \sin \theta. \tag{60}
\]

Since there are no losses due to friction or wind resistance, energy is conserved. We can therefore equate the old energy (all potential) with the new total energy, and then solve for \(\dot{\theta}\),

\[
\mathcal{K}(0^\circ, \dot{\theta}) + \mathcal{P}(0^\circ) = \mathcal{P}(30^\circ)
\]

\[
\frac{1}{2} (m_1 + m_2) \dot{\theta}^2 + (m_2 - m_1) g \sin(0) = (m_2 - m_1) g \sin(30) \tag{62}
\]

\[
\dot{\theta} = \pm \sqrt{\frac{(m_2 - m_1) g}{m_2 + m_1}} = \pm \sqrt{\frac{g}{3}}. \tag{63}
\]

Since we know that the seesaw is accelerating in a clockwise direction and started from rest, we select \(\dot{\theta} = -\sqrt{\frac{g}{3}} \approx -1.8\) rad/sec.

3. Now, supposing that the heavier mass falls off the seesaw, our intuition tells us that it should begin to accelerate counterclockwise.

\[f_f = (m_1 + m_2) g\]
\[r_1 = 1\ m\]
\[r_2 = 1\ m\]
\[h = 0.5\ m\]
\[m_1 = 20\ kg\]
\[f_i = -m_1 g\]

Setting \(m_2 = 0\), we may retain our earlier equations of motion from (57) as

\[
\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial}{\partial \theta} \left( \frac{1}{2} (m_1 + m_2) \dot{\theta}^2 - (m_2 - m_1) g \sin \theta \right) \tag{64}
\]

\[
= (m_1 + m_2) \dot{\theta} = m_1 \dot{\theta} \tag{65}
\]
\[ \frac{\partial L}{\partial q} = \frac{\partial}{\partial \theta} \left( \frac{1}{2} (m_1 + m_2) \dot{\theta}^2 - (m_2 - m_1)g \sin \theta \right) \] 
\[ = - (m_2 - m_1)g \cos \theta = m_1 g \cos \theta \]  
\[ (66) \]

\[ \tau = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \] 
\[ = \frac{d}{dt} m_1 \dot{\theta} - m_1 g \cos \theta \] 
\[ = m_1 \ddot{\theta} - m_1 g \cos \theta. \]  
\[ (67) \]

Now we can compute the equation of motion

\[ \ddot{\theta} = \frac{\tau}{m_1} + g \cos \theta \]  
\[ (71) \]

Here, \( \tau \) represents other forces not yet accounted for. This would include losses due to friction and wind resistance. If the seesaw were a robot, the motor torque at the joint would also be included here. As it is, there are no additional forces, so we can set \( \tau = 0 \), meaning that the acceleration is simply \( \ddot{\theta} = g \cos \theta = g \text{ rad/sec}^2 \) at the horizontal configuration.

4. Finally, at what angle will the seesaw reach zero velocity? To find the answer, we integrate both sides of (71) with \( \tau = 0 \) to get

\[ \dot{\theta} = \int_0^{t_f} \frac{\tau}{m_1} + g \cos \theta dt \] 
\[ = g \sin \theta \dot{\theta} - g \sin(0) \] 
\[ = g \sin \theta f \]  
\[ (72) \]

Now, set the final velocity

We again use the total energy at two points in time. At the horizontal configuration (time 1), we have

\[ \mathcal{K}(\theta_1, \dot{\theta}_1) = \frac{1}{2} m_1 \dot{\theta}_1^2 \] 
\[ \mathcal{P}(\theta_1) = 0 \] 
\[ (75) \]

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At the zero-velocity configuration (time 2), we have

\[ K(\theta_2, \dot{\theta}_2) = 0 \]  \tag{77}

\[ \mathcal{P}(\theta_2) = m_1 g \sin \theta_2 \]  \tag{78}

Once again employing conservation of energy, we set the sums equal and substitute known values,

\[ K(\theta_1, \dot{\theta}_1) + \mathcal{P}(\theta_1) = K(\theta_2, \dot{\theta}_2) + \mathcal{P}(\theta_2) \]  \tag{79}

\[ -\frac{1}{2} m_1 \frac{g}{3} + 0 = 0 + m_1 g \sin \theta_2 \]  \tag{80}

\[ \theta_2 = \sin^{-1} \left( -\frac{\frac{1}{2} m_1 \frac{g}{3}}{m_1 g} \right) = \sin^{-1} \left( -\frac{1}{6} \right). \]  \tag{81}

References


