Control*

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*These notes are mainly based on [1], [2] and [3].
1 Definitions

In this section, we provide basic definitions from [3] that will be used in the remainder of this chapter.

**System**: A combination of components that act together and perform a certain objective.

**System State**: A collection of variables that describe the current configuration of the system.

**Control**: The operation of measuring the value of a system variable and applying an action to correct or limit its deviation from a desired value.

**Controlled Variable**: The system variable being measured and controlled.

**Control Signal**: The action applied to affect the value of the controlled variable towards a desired reference value.

**Disturbance**: A signal that tends to adversely affect the value of the output of the system.

**Closed-Loop/Feedback Control**: Feedback Control or Closed-Loop Control refers to an operation that, in the presence of disturbances, tends to maintain a prescribed relationship between the output and a reference input by comparing them and using the difference as a means of control. Fig. 1 depicts the block diagram of a typical closed loop control architecture. The input to the system is a reference signal $r$, whose value is compared with the measured value of the output $y + n$, where $y$ is the actual output and $n$ is measurement noise. This comparison produces an error signal $e = r - (y + n)$, which is fed to the controller to produce a control signal $u$ which interacts with the system, upon incorporating a disturbance $d$.

**Open-Loop Control**: A control action that does not take into consideration the output of the system.

**Open-Loop versus Closed-Loop Control**: The closed loop design offers the ability to reject unpredictable disturbances and reduce the effect of internal variations in system parameters. Therefore, even with relatively inexpensive and inaccurate system components, it may be possible to achieve the desired system performance. Doing so with open-loop design is impossible. However, the stability of the system is an important problem that needs to be taken into consideration when designing the control strategy, whereas in open-loop design, stability is not an issue. Furthermore, open-loop design offers the advantages of simplicity, ease of maintenance, lower cost and lower power consumption, as less components are generally required. In cases where disturbances can be known a priori, open-loop design is thus preferred.
2 The State Space Model

Let $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ be vectors containing the state variables, control variables and measured signal of a system respectively. Then the dynamics of the system can be modeled with the following system of differential equations:

$$\frac{dx}{dt} = f(x, u), \quad y = h(x, u)$$  \hspace{1cm} (1)

where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^q$.

A model of this form is called a state space model. The dimension of the state vector is called the order of the system. The functions $f, h$ do not depend explicitly on time $t$ and for this reason the system 1 is called time-invariant.

If the functions $f, h$ are linear in $x$ and $u$, the system is called linear time-invariant (LTI) and can be represented as:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$  \hspace{1cm} (2)

with $A, B, C, D$ being constant matrices. $A$ is called the dynamics matrix, $B$ is called the control matrix, $C$ is called the sensor matrix and $D$ is called the direct term.

Another way of representing the dynamics of a system is through the following $n$th order differential equation:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = u$$  \hspace{1cm} (3)

where $t$ is the independent variable, $y(t)$ is the dependent (output) variable and $u(t)$ is the input. The notation $\frac{d^k y}{dt^k}$ represents the $k$th derivative of $y$ with respect to $t$. The system of eq. 3 can be converted to the state space form by considering the following definitions:
\[
x = \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \end{bmatrix}^T = \begin{bmatrix} \frac{d^{n-1} y}{dt^{n-1}} & \frac{d^{n-2} y}{dt^{n-2}} & \cdots & \frac{d y}{d t} & y \end{bmatrix}^T
\] (4)

Then, from eq. 3, we can form the following system of equations:

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\
x_2 \\
\vdots \\
x_{n-1} \\
x_n \end{bmatrix} = \begin{bmatrix} -a_1 x_1 - \cdots - a_n x_n \\
x_1 \\
\vdots \\
x_{n-2} \\
x_{n-1} \end{bmatrix} + \begin{bmatrix} u \\
0 \\
\vdots \\
0 \\
0 \end{bmatrix}, \quad y = x_n
\] (5)

Finally, if we assume that the output is a linear combination of the states and the control input, i.e.,

\[
y = b_1 x_1 + b_2 x_2 + \cdots + b_n x_n + du
\] (6)

then, the system can be written in the following general form:

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\
x_2 \\
\vdots \\
x_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\
1 \\
0 \\
\vdots \\
0 \end{bmatrix} u
\] (7)

\[
y = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} x + du.
\] (8)

This form is called the Reachable Canonical Form.

### 2.1 System Response

The general solution to a linear ordinary differential equation of the form

\[
\frac{d^n y}{dt^n} = \sum_{i=0}^{n-1} a_i(x) \frac{d^i y}{dt^i} + r(x)
\] (9)

is given by the equation

\[
y = y_c + y_p
\] (10)

where \(y_c\) is the solution to the corresponding homogeneous equation, i.e., eq. 9 with \(r(x) = 0\), and \(y_p\) is an additional solution accounting for the term \(r(x)\). In the case of the state space system (eq. 2), the general solution corresponds to the system response.
2.1.1 Initial Condition Response

Consider the homogeneous response, corresponding to the system:

\[
\frac{dx}{dt} = Ax
\]  

(11)

For a scalar equation of the same form, i.e.,

\[
\frac{dx}{dt} = ax, \quad x, a \in \mathbb{R},
\]  

(12)

the solution is given by:

\[x(t) = e^{at}x(0).\]  

(13)

This form of solution can be generalized for the case of \(A\) being a matrix through the use of the matrix exponential. The matrix exponential can be written as an infinite series:

\[e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}X^k\]  

(14)

where \(X \in \mathbb{R}^{n \times n}\) and \(I\) is the \(n \times n\) identity matrix.

Substituting \(X\) with \(At\) in eq. 14, we get:

\[e^{At} = I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k\]  

(15)

which, upon differentiation with respect to \(t\), gives:

\[\frac{d}{dt} e^{At} = A + A^2t + \frac{1}{2}A^3t^2 + \cdots \]  

(16)

\[= A \left( I + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \right) \]  

(17)

\[= A \sum_{k=0}^{\infty} \frac{1}{k!}A^k t^k = Ae^{At}.\]  

(18)

If we postmultiply eq. 18 by \(x(0)\), we get:

\[\frac{d}{dt} (e^{At}x(0)) = A (e^{At}x(0))\]  

(19)

and thus we conclude that

\[x(t) = e^{At}x(0)\]  

(20)

is a solution to the system of equations of eq. 11. Therefore, eq. 20 gives the response of the system to an initial condition \(x(0)\), under no control input \(u\).
2.1.2 Input/Output Response

We wish to derive the system response under the application of a control input \( u(t) \). This corresponds to the solution of the system:

\[
\frac{dx}{dt} = Ax + Bu, \quad y = Cx + Du \tag{21}
\]

which can be shown (validate this by differentiating both sides and employing the property of eq. 18) to be:

\[
x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{22}
\]

and thus it follows that:

\[
y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{23}
\]

3 Stability

One of the central topics of interest in the study of dynamical systems (such as the state-space models we discussed in the previous section) is stability. Intuitively, given a system with dynamics described by a set of differential equations, stability addresses the problem of determining the system’s behavior, under small perturbations of its initial conditions.

A solution to a differential equation \( \dot{x} = f(x) \) with initial condition \( a \), \( x(t; a) \), is stable if other solutions that start near \( a \) stay close to \( x(t; a) \). Formally, the solution \( x(t; a) \) is stable if for all \( \epsilon > 0 \), there exists a \( \delta > 0 \), such that:

\[
\|b - a\| \leq \delta \implies \|x(t; b) - x(t; a)\| < \epsilon \text{ for all } t > 0. \tag{24}
\]

This notion of stability is referred to as Stability in the sense of Lyapunov. An interpretation of this definition is that starting sufficiently close to a certain initial condition, we are guaranteed to stay close to its corresponding solution. If a solution is stable in this sense of Lyapunov but trajectories starting from different initial conditions do not converge, then the solution is called neutrally stable.

An equilibrium point \( x_e \) of a dynamical system represents a stationary condition for its dynamics. Formally, \( x_e \) is an equilibrium point for a dynamical system \( \frac{dx}{dt} = f(x) \) if \( f(x_e) = 0 \). This implies that a system...
with an initial condition \( x(0) = x_e \) will stay at the equilibrium point, i.e., \( x(t) = x_e \) for all \( t \geq 0 \). In general, a dynamical system can have zero, one or more equilibrium points. When a solution \( x(t; a) = x_e \) is an equilibrium solution, we say that the equilibrium point is stable.

A solution \( x(t; a) \) is called \textit{asymptotically stable} if it is stable in the sense of Lyapunov and also satisfies

\[
x(t; b) \to x(t; a) \text{ as } t \to \infty \text{ for } b \text{ sufficiently close to } a.
\] (25)

This is the case where all nearby trajectories converge to the stable solution as time goes to infinity. Specifically for planar systems, asymptotically stable equilibrium points are commonly referred to as \textit{attractors}, whereas equilibrium points that is stable but not asymptotically stable are called \textit{centers}.

A solution is \textit{locally stable} or \textit{locally asymptotically stable} if it is stable for all initial conditions \( x \in B_r(a) \), where \( B_r(a) = \{ x : \| x - a \| < r \} \) is a ball of radius \( r \) around \( a \) and \( r > 0 \). A system is globally stable if it is stable for all \( r > 0 \).

A solution \( x(t; a) \) is unstable if it is \textit{not stable}. Formally, a solution \( x(t; a) \) is unstable if given some \( \epsilon > 0 \), there does not exist a \( \delta > 0 \) such that if \( \| b - a \| < \delta \), then \( \| x(t; b) - x(t; a) \| < \epsilon \) for all \( t \). An unstable equilibrium point of a planar system is typically referred to as a \textit{source}, if all trajectories move away from the equilibrium point, or \textit{saddle}, if some trajectories lead to the equilibrium point and others move away.

\subsection*{3.1 Stability of Linear Systems}

For a linear dynamical system described by dynamics of the form

\[
\dot{x} = Ax, \quad x(0) = x_0
\] (26)

the stability of the equilibrium at the origin (notice that the origin is always an equilibrium point for such a system) can be examined by looking at the eigenvalues of matrix \( A \):

\[
\lambda(A) = \{ s \in \mathbb{C} : \det(sI - A) = 0 \}
\] (27)

where \( \mathbb{C} \) is the set of complex numbers, the polynomial \( p_A = \det(sI - A) \) is the \textit{characteristic polynomial} of matrix \( A \) and its eigenvalues are the roots of the equation \( p_A = 0 \). Observe that here, stability is a property of the system, since it depends on the matrix \( A \).
In general, it can be shown that a linear system $\frac{dx}{dt} = Ax$ is asymptotically stable if and only if all eigenvalues of $A$ all have a strictly negative real part and is unstable if any eigenvalue of $A$ has a strictly positive real part.

### 3.2 Lyapunov Stability Analysis

For the general case of a (possibly nonlinear) system

$$\frac{dx}{dt} = f(x), \ x \in \mathbb{R}^n$$

we are interested in proving that a given solution is stable, asymptotically stable or unstable. Lyapunov Stability Analysis provides a formal tool for doing so, based on the definition of energy-like functions, called the Lyapunov functions.

Consider a nonnegative function $V: \mathbb{R}^n \rightarrow \mathbb{R}^+$, that always decreases along system trajectories, thus its minimum is a locally stable equilibrium point. Therefore, if we find a function with those properties for a given system, we can prove that the system is stable. Such a function is called a Lyapunov function. To characterize the stability of a system in the sense of Lyapunov, we make use of the Lyapunov Stability Theorem. To formulate the theorem, we first need to go over a few definitions which it makes use of:

- A continuous function $V$ is positive definite if $V(x) > 0$ for all $x \neq 0$ and $V(0) = 0$.
- A continuous function $V$ is negative definite if $V(x) < 0$ for all $x \neq 0$ and $V(0) = 0$.
- A continuous function $V$ is positive semidefinite if $V(x) \geq 0$ for all $x$, but $V(x)$ can be zero at points other than just $x = 0$.

### Lyapunov Stability Theorem

Let $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$ and denote by $\dot{V}$ its time derivative along trajectories of system dynamics:

$$\dot{V} = \frac{\partial V}{\partial x} \frac{dx}{dt} = \frac{\partial V}{\partial x} f(x).$$

Let also $B_r = B_r(0)$ be a ball of radius $r$ around the origin. If there exists $r > 0$ such that $V$ is positive definite and $\dot{V}$ is negative semidefinite for all
$x \in B_r$, then $x = 0$ is *locally stable in the sense of Lyapunov*. If $V$ is positive definite and $\dot{V}$ is negative definite in $B_r$, then $x = 0$ is *locally asymptotically stable*.

**Example 1**

Consider a simple mass-spring system, as shown in figure ???, and suppose friction force is proportional to the block’s velocity. Then we can write the motion equation as

$$m\ddot{x} + b\dot{x} + kx = 0 \quad (30)$$

Then we can get the total energy of the system as

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \quad (31)$$

Then we can find the change rate of the energy of the system in equation 31 as

$$\dot{E} = m\dot{x} \ddot{x} + kx \dot{x} \quad (32)$$

Put equation 30 into equation 32, we can obtain:

$$\dot{\dot{v}} = -b\dot{x}^2 \quad (33)$$

Therefore, the energy of the system always leaves the system unless $\dot{x} = 0$, which implies that the system will release the energy until it comes to rest regardless of initial state. Hence, we have shown that this spring-mass system will eventually come to rest at the equilibrium.

![Simple spring—mass system with friction coefficient b](image)
**Example 2**

Consider a manipulator with dynamics

\[ \tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta) \]  

(34)

and controlled law

\[ \tau = K_pE - K_d\dot{\Theta} + G(\Theta) \]  

(35)

where \( K_p, K_d \) are diagonal gain matrices. Then we can get close-loop system as

\[ M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + K_d\dot{\Theta} + K_p\Theta = K_p\Theta \]  

(36)

Then we need to consider the candidate Lyapunov function:

\[ E_v = \frac{1}{2}\dot{\Theta}^T M(\Theta) \dot{\Theta} + \frac{1}{2}E^T K_p E \]  

(37)

Then we need to compute the derivative of the system

\[ \dot{E}_v = \frac{1}{2}\dot{\Theta}^T M(\Theta) \dot{\Theta} + \dot{\Theta}^T M(\dot{\Theta}) \dot{\Theta} - E^T K_p \dot{\Theta} = -\dot{\Theta}^T K_d \dot{\Theta} \]  

(38)

We can see that as long as \( K_d \succ 0 \), the change rate of the system is nonpositive. In addition, \( \dot{E}_v \) can be zero only along the trajectories where \( \dot{\Theta} = 0, \ddot{\Theta} = 0 \), we can get \( K_pE = 0 \) and thus, \( E = 0 \). Therefore, equation 35 in system 34 have global asymptotic stability.

4 State Feedback

4.1 Why feedback?

Why should one use feedback? What are the advantages (and disadvantages) of feedback control over other control architectures? In practice, there are many tradeoffs (including weight, added complexity, reliability of sensors, etc), but there are two fundamental principles of feedback.

Modify Dynamics

Feedback can modify the natural dynamics of a system. For instance, using feedback, one can improve the damping of an underdamped system, or stabilize an unstable operating condition, such as balancing an inverted pendulum. However, open-loop or feed-forward approaches cannot do this.
Reduce sensitivity

Feedback can also reduce sensitivity to external disturbances, or to changing parameters in the system itself (the plant). For instance, an automobile with a cruise control that senses the current speed can maintain the set speed in the presence of disturbances such as hills, headwinds, etc. In this example, feedback also compensates for changes in the system itself, such as changing weight of the vehicle as the number of passengers changes. If instead of using feedback, a lookup table were used to select the appropriate throttle setting based only on the desired speed, such an open-loop system would not be able to compensate either for disturbances or changes in the vehicle itself.

4.2 Reachability and Controllability

Informally, the concepts of Controllability and Reachability examine respectively whether a given state can be driven to the origin and whether a given state can be reached from the origin. Formally, for a system with dynamics:

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  

(39)

a state \( x_d \neq 0 \) is said to be reachable in time \( T \) if there exists a finite time interval \([0, T]\) and an input \( u : [0, T] \rightarrow \mathbb{R}^p \) that can drive the system from \( x_0 = 0 \) to \( x_d \) in time \( T \), whereas a state \( x_0 \) is said to be controllable in time \( T \) if there exists a finite time interval \([0, T]\) and an input \( u : t \in [0, T] \rightarrow \mathbb{R}^p \) that can drive the system from \( x_0 \) to zero in time \( T \).

Let \( \mathcal{R} \) be the set of all reachable points, i.e., the set of all points in \( \mathbb{R}^n \) that can be reached within a finite amount of time, starting from \( x_0 = 0 \). The set \( \mathcal{R} \) is a linear subspace, i.e., if \( x_1, x_2 \in \mathbb{R}^n \) are reachable states, so is any linear combination of them. If the reachable set is the entire state space, i.e., if \( \mathcal{R} = \mathbb{R}^n \), then the system is called (completely) reachable. Likewise, if all states are controllable, then the system is called (completely) controllable.

For continuous, linear time-invariant systems, a system is controllable if and only if it is reachable, i.e., controllability implies reachability and vice versa.

4.3 Testing for Reachability/Controllability

Consider an impulse input defined as

\[ \delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t) \]  

(40)
for
\[ p_\epsilon(t) = \begin{cases} 
0 & t < 0 \\
1/\epsilon & 0 \leq t < \epsilon \\
0 & t \geq 0 
\end{cases}, \tag{41} \]
acting on a system with zero initial state. The system response can then be found to be:
\[ x_\delta = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau = e^{At}B \tag{42} \]
The response to the derivative of an impulse response can be shown to be given by the taking the derivative of the impulse response, i.e.,
\[ x_\delta = \frac{dx_\delta}{dt} = Ae^{At}B. \tag{43} \]
Taking the higher derivatives of the impulse responses, we can construct the input
\[ u(t) = a_1\delta(t) + a_2\delta'(t) + \cdots + a_n\delta^{(n-1)}(t) \tag{44} \]
which gives the expression for the corresponding response
\[ x(t) = a_1e^{At}B + a_2Ae^{At}B + \cdots + a_nA^{n-1}e^{At}B. \tag{45} \]
Taking the limit as \( t \) goes to zero, we get the response
\[ \lim_{t \to 0^+} x(t) = a_1B + a_2AB + \cdots + a_nA^{n-1}B \tag{46} \]
which is a linear combination of the columns of the matrix
\[ C = [B \ AB \ \cdots \ A^{n-1}B]. \tag{47} \]
This matrix is called the \textit{Reachability/Controllability} Matrix. In order to reach an arbitrary point in the state space, the reachability matrix needs to have \( n \) linearly independent columns, or, equivalently, \( C \) needs to be of rank \( n \). This result can be generalized for different (smoother) signals.

4.4 Observability

Given an input-output system, we have known that Controllability is used to describe the ability of the actuator \( f \) to influence the state \( q \). Conversely, observability describes the ability to reconstruct the full state \( q \) from available sensor measurements \( y \). For instance, in a fluid flow, often we cannot measure the entire flow-field directly. The sensor measurements \( y \) might
consist of a few pressure measurements on the surface of a wing, and we may be interested in reconstructing the flow everywhere in the vicinity of the wing. Observability describes whether this is possible.

Suppose we have linear systems

\[ \dot{q}(t) = Aq(t) + Bf(t) \quad (48) \]
\[ y(t) = Cq(t) \]

The system above is said to be observable if for all initial times \( t_0 \), the state \( q(t_0) \) can be determined from the output \( y(t) \) defined over a finite time interval \( t \in [t_0, t_1] \). The important point here is that we do not simply consider the output at a single time instant, but rather watch the output over a time interval, so that at least a short time history is available.

4.5 Stabilization by State Feedback

Fig. 3 depicts the block diagram of a system, controlled with state feedback. The controller comprises two elements \( k_r \) and \( K \), with the former acting on the reference \( r \) and the latter acting on the state \( x \). The input to the system is determined by the control law

\[ u = k_r r - K x \quad (49) \]

and the disturbance \( d \). Thus the controller adjusts the system dynamics as:

\[ \frac{dx}{dt} = (A - BK)x + Bk_r r. \quad (50) \]

The system stability, as well as the transient response of the controlled system are determined by the eigenvalues of the closed-loop state matrix \( A_{cl} = A - BK \). Therefore, by picking \( K \), we manipulate the eigenvalues to achieve desired properties. In particular, to pick \( K \), we typically write down a parametric expression of the characteristic equation of \( A_{cl} \):

\[ p_{A_{cl}}(s) = \det(sI - A_{cl}) = 0 \quad (51) \]

and select the entries of \( K \) matrix to place its roots (the eigenvalues of \( A_{cl} \), also called poles) to be such that desired performance specifications are achieved.

On the other hand, \( k_r \) only affects the steady state response. This can be seen by computing the equilibrium point \( x_e \) (by setting \( dx/dt = 0 \)) and the steady-state output for the closed loop system, i.e.,

\[ x_e = -A_{cl}^{-1}Bk_r r, \quad y_e = Cx_e + Du_e. \quad (52) \]
\[ x' = Ax + Bu \]
\[ y = Cx + Du \]

Controller Process

\[ \Sigma \]
\[ \Sigma \]
\[ r \]
\[ k_r \]
\[ d \]
\[ \Sigma \]
\[ u \]
\[ y \]
\[ x \]
\[ y_r \]
\[ x \]
\[ u \]

Figure 3: A block diagram of a system with state feedback.

Substituting \( x_e \) in the output expression and assuming that \( D = 0 \), we find the value of \( k_r \) that ensures that \( y_e = r \):

\[ k_r = -\frac{1}{CA^{-1}B} \]  \hfill (53)

4.5.1 Ackermann’s Formula

For a controllable system, Ackermann’s formula provides a method for determining the gain matrix \( K \):

\[ K = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} C^{-1}p_{A_{cl}}(A) \]  \hfill (54)

where \( C \) is the system’s controllability matrix and \( p_{A_{cl}}(A) \) is the closed loop characteristic polynomial, evaluated for \( s = A \). The controllability requirement can be understood from the inversion of the controllability matrix.

4.5.2 The Linear Quadratic Regulator

Another widely used method of determining the gain matrix \( K \) is by employing optimal control techniques, i.e., by formulating optimization problems that represent certain desired performance specifications. For a system with dynamics modeled by the system of eq. 2, the infinite-horizon, linear quadratic regulator (LQR) problem can be defined as to find the control \( f(t) \) that minimize the following quadratic objective function:

\[
\tilde{J}(f) = \frac{1}{2} \int_0^\infty (q(t)^T Q q(t) + f(t)^T R f(t)) dt
\]  \hfill (55)
subject to the state dynamics $\dot{\mathbf{q}}(t) = A\mathbf{q}(t) + B\mathbf{f}(t)$, $\mathbf{q}(0) = \mathbf{q}_0$.

where $Q, R \geq 0$ are symmetric, positive semi-definite matrices of appropriate dimensions. More specifically, the matrices $Q$ and $R$ are state and control weighting operators and may be chosen to obtain desired properties of the closed loop system. This cost represents a compromise between the distance of the state from the origin (first term) and the energy required to generate the control input (second term). This compromise expresses a balance between the rate of convergence to the desired state with the control cost. This problem is called linear because the dynamic constraints are linear, and quadratic since the objective function is quadratic. The controller is called a regulator because the optimal control will drive the state to zero.

The solution to the LQR problem can be obtained by applying the necessary conditions of the calculus of variations. We can find that under the above assumptions, the solution to this problem is given by the feedback control

$$\mathbf{f}(t) = -R^{-1}B^T\mathbf{P}\mathbf{q}(t)$$

where $\mathbf{P} \in \mathbb{R}^{n \times n} \geq 0$ is a symmetric matrix that satisfies the algebraic Riccati equation:

$$PA + A^TP - PBR^{-1}B^TP + Q = 0.$$  \hfill (57)

This is a quadratic matrix equation, and has many solutions $\mathbf{P}$, but only one positive-definite solution, which is the one we desire.

### 4.5.3 Long-Form Example

**Example 1** (Balancing a Broom).

**Concepts reviewed:** Euler-Lagrange equations of motion, linear equations of motion, linear quadratic regulator

**Problem:** Consider a system in which a hand moves side-to-side in order to maintain balance for an inverted broom. You may regard the broom as a point mass $m$ located at a distance $\ell$ vertically away from the palm of the hand. The hand has mass $M$ and may only move sideways – not up or down. The broom handle cannot slip or separate from the hand. Design an LQR controller to balance the broom while limiting horizontal force $F$ on the hand.

Suppose $\ell = 1$ m, $m = 1$ kg, $M = 1$ kg

**Solution:** We begin by identifying the configuration of the system in two variables: $\theta$ is the angle of the broom, and $y$ is the $y$-position of the hand’s
Figure 4: A hand (beige box) is supporting a broom and trying to balance it. The problem is 2D, and there are also two degrees of freedom – the tilting of the broom and the sideways motion of the hand. The broom’s angle may only be indirectly actuated by sliding the hand to move more underneath the point center of mass of the broom.

coordinate frame in the global frame. Together with rates, the state vector and control vector respectively are

\[
x = \begin{bmatrix} \theta \\ \dot{\theta} \\ y \\ \dot{y} \end{bmatrix}
\]

\[u = \begin{bmatrix} F \end{bmatrix}.
\]

We proceed by computing the system dynamics using the Euler-Lagrange equations of motion. First, we need to compute potential and kinetic energy. In calculating potential energy, we need only consider the mass of the broom because the mass of the hand is located at the height of the origin and never changes,

\[P = mgh = mgl \cos \theta.
\]

Kinetic energy stems from the momentum of both the broom’s and hand’s masses. The contribution of angular velocity from the broom can be neglected because we are instructed to treat it as a point mass. Thus, we need
only linear velocity. The instantaneous linear velocity of the broom’s mass is

\[ v = \begin{bmatrix} 0 \\ \dot{y} \end{bmatrix} + \ell \ddot{\theta} \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} = \begin{bmatrix} -\ell \dot{\theta} \sin \theta \\ \dot{y} + \ell \dot{\theta} \cos \theta \end{bmatrix}. \quad (61) \]

Now we are prepared to write the kinetic energy,

\[ K = \frac{1}{2} m v^2 = \frac{1}{2} m (\ell \ddot{\theta}^2 \sin^2 \theta + \dot{y}^2 + 2 \ell \dot{\theta} \dot{y} \cos \theta + \ell^2 \dot{\theta}^2 \cos^2 \theta) + \frac{1}{2} M \dot{y}^2 \]

\[ = \frac{1}{2} m (\ell^2 \dot{\theta}^2 + \dot{y}^2 + 2 \ell \dot{\theta} \dot{y} \cos \theta) + \frac{1}{2} M \dot{y}^2, \quad (62) \]

with terms for the broom and hand masses, respectively. Now we can write the Lagrangian and differentiate it,

\[ L = K - P = \frac{1}{2} m (\ell^2 \dot{\theta}^2 + \dot{y}^2 + 2 \ell \dot{\theta} \dot{y} \cos \theta) + \frac{1}{2} M \dot{y}^2 - m g l \cos \theta \]

\[ = \ell^2 \dddot{\theta} - \ell \dot{\theta} \dot{y} \sin \theta \]

\[ = m \dddot{y} + \ell \dddot{\theta} \cos \theta - M \dddot{y} \]

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{y}} = m \dddot{y} + \ell \dddot{\theta} \cos \theta - \ell \dot{\theta} \dot{y} \sin \theta + M \dddot{y} \]

\[ \frac{\partial L}{\partial y} = 0. \quad (71) \]

Now we can write the Euler-Lagrange equations,

\[ \tau = 0 = \ell^2 m \dddot{\theta} + \ell m \dddot{y} \cos \theta - \ell m \dot{\theta} \dot{y} \sin \theta - g \ell m \sin \theta \]

\[ = \ell^2 m \dddot{\theta} + \ell m \dddot{y} \cos \theta - g \ell m \sin \theta \]

\[ u = F = m \dddot{y} + \ell m \dddot{\theta} \cos \theta - \ell m \dot{\theta}^2 \sin \theta + M \dddot{y}. \quad (74) \]

At this point, we can proceed to linearize the dynamics equations by approximating \( \theta = 0, \dot{\theta} = 0, \sin \theta = \theta, \cos \theta = 1 \),

\[ 0 = \ell^2 m \dddot{\theta} + \ell m \dddot{y} - g \ell m \theta \]

\[ u = m \dddot{y} + \ell m \dddot{\theta} + M \dddot{y} \]
Finally, our goal is to write out the $A$ and $B$ matrices to conform to the linear dynamics equation (39). To do this, we must solve equations (75) and (76) for $\ddot{\theta}, \ddot{y},$

\[
\ddot{\theta} = \frac{g(m + M)}{\ell M} \theta - \frac{1}{\ell M} u, \tag{77}
\]

\[
\ddot{y} = \frac{g(m + 3M)}{M} \theta + \frac{1}{M} u, \tag{78}
\]

yielding (and substituting)

\[A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\frac{g(m+M)}{\ell M} & 0 & 0 & 0 \\
\frac{g(M-m)}{M} & 0 & 0 & 1 \\
\frac{1}{\ell M} & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
19.6 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}, \tag{79}
\]

\[B = \begin{bmatrix}
0 \\
-\frac{1}{\ell M} \\
0 \\
\frac{1}{M}
\end{bmatrix} = \begin{bmatrix}
0 \\
-1 \\
0 \\
1
\end{bmatrix}. \tag{80}
\]

Now, we write the cost expression that will be optimized by our controller. We select an infinite-horizon cost,

\[J = \int x^T Q x + u^T R u \, dt \tag{81}\]

We assign values of $Q$ and $R$ that reflect our disinterest in the position at which the broom is balanced,

\[Q = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 10^{-8} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \tag{82}\]

\[R = [1]. \tag{83}\]

The LQR optimal controller will be of the form

\[u = -K x, \tag{84}\]

with

\[K = R^{-1} B^T P, \tag{85}\]
and $P$ is the solution to the Algebraic Riccati Equation (algebraic because it is an infinite horizon problem),

$$0 = A^T P + PA - PBR^{-1}B^T P + Q. \quad (86)$$

For convenience, we plug these equations into a numerical solver,

$$P = \begin{bmatrix} 262.9717 & 59.0240 & 0.0011 & 10.9123 \\ 59.0240 & 13.3659 & 0.0002 & 2.4550 \\ 0.0011 & 0.0002 & 0.0001 & 0.0001 \\ 10.9123 & 2.4550 & 0.0001 & 1.4548 \end{bmatrix}. \quad (87)$$

$$K = \begin{bmatrix} -48.1118 \\ -10.9109 \\ -0.0001 \\ -1.0001 \end{bmatrix}. \quad (88)$$

5 Output Feedback

So far, we have made the assumption that all states included in $x$ can be directly measured. In reality, this is not always the case. In this section, we show how we can estimate the states, through a mathematical model that makes use of output measurements and the applied input. The underlying computations can be implemented with a dynamical system, called observer.

Consider again the state-space system with dynamics described by eq. 2. The output measurement, which in general incorporates measurement noise $n$, is then fed to the observer, which outputs a state estimate $\hat{x}$, as shown in Fig. 5. To characterize a system with respect whether it is possible to estimate its state from measurements of the input and output, introduce the notion of Observability. A linear system is Observable if for any $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$. 

Figure 5: Block diagram of a system with an observer.
5.1 Testing for Observability

When testing for Reachability, in the previous section, we neglected the output. Likewise, in this section, we will be neglecting the effect of the input for now, i.e., we will be considering the system dynamics:

\[ \dot{x} = Ax, \quad y = Cx. \]  

Notice that if the matrix \( C \) is invertible, the observability problem can be directly solved by inverting \( C \). To account for the general case, in which \( C \) might not be invertible, we take the following approach. We take the output derivatives with respect to time. For the first derivative, we have:

\[ \frac{dy}{dt} = \frac{dx}{dt} = CAx \]

and we continue for the higher derivatives to form the following matrix form:

\[
\begin{bmatrix}
y \\
dot{y} \\
\ddot{y} \\
\vdots \\
y^{(n-1)}
\end{bmatrix} =
\begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}.
\]

The matrix \( O = \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix} \) is called the Observability matrix. It can be observed that the state can be determined if the matrix \( O \) has \( n \) independent rows or if \( O \) is full rank. It can be shown, by application of the Cayley-Hamilton theorem, that no derivatives or order higher than \( n - 1 \) are needed. Note that this construction can be similarly implemented for systems with inputs; however the result -the observability criterion as formulated above- is unchanged.

Finally, note the the outlined method is not practically efficient, as the differentiation of the output can lead to large errors if the measurements are particularly noisy.
6 PID Control

The general form of a PID controller (PID stands for Proportional Integral Derivative action) can be defined as:

\[ u(t) = K \left( e(t) + \frac{1}{T_i} \int_0^t e(\tau)d\tau + T_d \frac{de(t)}{dt} \right) \]  \hspace{1cm} (92)

where

\[ e(t) = r - y(t) \] \hspace{1cm} (93)

is the error in the output, defined as the difference between the reference (set-point) \( r \) and the measured output \( y \). The control signal \( u(t) \) is the sum of three terms:

- the P-term which is proportional to error,
- the I-term which is proportional to the integral of the error
- and the D-term which is proportional to the derivative of the error.

The controller parameters corresponding to these three terms are: the proportional gain \( K \), the integral time \( T_i \) and the derivative time \( T_d \) respectively.

6.1 Effects of Proportional, Integral and Derivative Action

In the case of purely proportional control (setting \( T_i = \infty \) and \( T_d = 0 \) in eq. 92), there will always be a steady-state error (can be proven by taking the Final Value Theorem). Incorporating integral action (by decreasing the integral time \( T_i \)) leads to the disappearance of the steady state error. However, this also increases the system’s tendency for oscillations, which is undesirable. This motivates the introduction of the derivative term, that adds damping to the system as \( T_d \) increases, until a certain threshold determined by the dynamics of the system.

More specifically, to see the effect of PI and PD feedback on these systems, consider a first-order system of the form

\[ \dot{y} + ay = f \] \hspace{1cm} (94)

where \( f \) is the input and \( y \) is the output, and \( a \) is a parameter and suppose that the control objective is to alter the dynamics of 94. Hence, We can also write PID controller as

\[ f(t) = -K_p y(t) - K_i \int_0^\infty y(\tau)d\tau - K_d \frac{dy(t)}{dt} \] \hspace{1cm} (95)
Without feedback (f=0), the system has a pole at -a, or a dynamic response of $e^{-at}$. We may use feedback to alter the position of this pole. Choosing PI feedback $f= -K_p y - K_i \int y dt$, the closed-loop system becomes

$$\ddot{y} + (a + K_p)\dot{y} + K_i y = 0$$  \hspace{1cm} (96)

The closed-loop system is now second order, and with this feedback, the poles are the roots of

$$s^2 + (a + K_p)s + K_i = 0$$  \hspace{1cm} (97)

Clearly, by appropriate choice of Kp and Ki, we may place the two poles anywhere we desire in the complex plane, since we have complete freedom over both coefficients in 97.

Next, consider a second-order system, a spring-mass system with equations of motion

$$m \ddot{y} + b \dot{y} + ky = f$$  \hspace{1cm} (98)

where m is the mass, b is the damping constant and k is the spring constant. With PD feedback, $u = -K_p y - K_d \dot{y}$, the closed-loop system becomes

$$m \ddot{y} + (b + K_d)\dot{y} + (k + K_p)y = 0$$  \hspace{1cm} (99)

with closed-loop poles satisfying

$$ms^2 + (b + K_d)s + k + K_p = 0.$$  \hspace{1cm} (100)

We may place the poles anywhere desired, since we have freedom to choose both of the coefficients in 100.

### 6.2 Transfer functions

Linear systems obey the principle of superposition, and hence many useful tools are available for their analysis. One of these tools that is ubiquitous in classical control is the Laplace transform. The Laplace transform of a function of time $y(t)$ is a new function $\tilde{y}(s)$ of a variable $s \in \mathbb{C}$, defined as

$$\tilde{y}(s) = L\{y(t)\} = \int_0^\infty e^{-st}y(t)dt$$  \hspace{1cm} (101)

defined for values of $s$ for which the integral converges. The most useful property of Laplace transformation is

$$L\left(\frac{dy}{dt}\right) = s\tilde{y}(s) - y(0)$$  \hspace{1cm} (102)
Hence, the Laplace transform converts differential equations into algebraic equations. For a plant with input $u$ and output $y$ and dynamics given by

$$a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{f} + b_0 f$$

(103)

taking the Laplace transform, with zero initial conditions, we can get

$$\tilde{y}(s) \frac{\tilde{u}(s)}{\tilde{u}(s)} = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} = P(s)$$

(104)

where $P(s)$ is called the transfer function from $f$ to $y$. Thus, the transfer function relates the Laplace transform of the input to the Laplace transform of the output.

References

