Problem 1: Probability

A (5 points): Consider a real-valued random variable $X$ that follows a uniform distribution between $a$ and $b$. What is the mean and standard deviation of $X$?

Solution: $(a + b)/2$ and $\sqrt{(a - b)^2/12}$

B (5 points): Consider a two-player dice game. On each round:

- Player 1 rolls a single 6-sided die and adds 1 to the result.
- Player 2 rolls 2 6-sided dice and takes the maximum of the two.
- Whoever has the lower number pays the other $1$. (In case of a tie no money is paid in either direction.)

Each player starts with $100$. After playing 100 times, what is the expected value of how much money each player winds up with?

Using a brute force method, enumerate the different rolls of Player 1 and then counting the expected number of wins vs losses out of the 36 possible roles of Player 2. The expected loss for Player 2/win for Player 1 winds up being about 0.00463, so multiple by 100 to get Player 1 is expect to win $100.46 and Player 2 is expected to get $99.54
C (5 points): Imagine that you have a bag containing 100 balls of which 20 are green and 80 are red. 90% of the green balls have a stripe, whereas 30% of the red balls are without a stripe. If you pull out one ball and it has a stripe, what is the probability that it is green?

\[
Pr(green|\text{stripe}) = \frac{Pr(\text{stripe}|green) \cdot Pr(green)}{Pr(\text{stripe})} = \frac{0.9 \cdot 0.2}{0.74} = 0.24
\]

\[
\text{Problem 2: Calculus}
\]

A (5 points): Consider the function \( f(x) = \frac{1}{1+e^{-w \cdot x}} \) where \( w = (w_0, ..., w_n) \) and \( x = (x_0, ..., x_n) \). What is \( \frac{\partial f}{\partial w_i}(x) \)?

\[
\frac{\partial f}{\partial w_i}(x) = \frac{x_i \cdot e^{-w \cdot x}}{(1 + e^{-w \cdot x})^2}
\]

B (5 points): For what value(s) of \( x \) is \( f(x, y) = 2x^2 + 4x + y^2 \) is \( f(x, y) \) minimized?

\( x = -1 \)

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\text{Problem 3: BST Traversals}
\]

Consider the following binary tree:

\[
\text{Problem 4: Complexity of BST}
\]

Consider a binary search tree \( T \) containing \( n \) nodes.

A (4 points): What is the worst-case number of comparisons that are made to answer whether a particular item is in \( T \)? Give the exact formula.

\[
\text{Solution: worst case } n
\]
B (2 points): Give the “big O” notation answer for part a. 
\[ O(n) \]

C (4 points): What is the worst-case number of comparisons that are made to answer whether a particular item is in T if T is balanced? Give the exact formula.

The depth of the tree will be \[ \lceil \log_2(n + 1) \rceil \], and this is the worst case number of comparisons.

D (2 points): Give the “big O” notation answer for part c. \[ O(\log_2(n)) \]

\begin{center}{\bf Problem 5: First Order Logic} (10 points)\end{center}

Answer True or False for the following statements:

A (2 points): False \( \models \) True
  (Vacuously) True

B (2 points): True \( \models \) False
  False

C (2 points): \[ [(A \land B) \Rightarrow C] \models [(A \Rightarrow C) \lor (B \Rightarrow C)] \]
  True

D (2 points): \[ [(A \lor B) \land (\neg C \lor \neg D \lor E)] \models (A \lor B) \]
  True

E (2 points): \[ [(A \lor B) \land \neg(A \Rightarrow B)] \] is satisfiable
  True (A True and B False)