CS 4700: Foundations of Artificial Intelligence

Fall 2019

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Lecture 19

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Elements of Formal Logic

- Syntax: What you can write down
- Semantics: The connection between what you write down and their meaning in the world being represented
- Inference: Making new conclusions based on what you already know

Propositional Logic: Syntax

- What is legal to write down in propositional logic:

Sentence → *AtomicSentence* | *ComplexSentence*
AtomicSentence → T | F | <PropositionalSymbol> /* string of alphanumerics */
ComplexSentence → ¬ *Sentence*
| *Sentence* ∨ *Sentence*
| *Sentence* ∧ *Sentence*
| *Sentence* ⇒ *Sentence*
| (*Sentence*)
| [*Sentence*]

- Operator order of precedence: ¬, ∧, ∨, ⇒ ¬ P ∨ Q vs ¬(P ∨ Q)

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Propositional Logic: Semantics

Truth Tables

φ	ψ	$\neg\varphi$	$\varphi \vee \psi$	$\varphi \wedge \psi$	$\varphi \Rightarrow \psi$
true	true	false	true	true	true
true	false	false	true	false	false
false	true	true	true	false	true
false	false	true	false	false	true

Propositional Logic: Semantics

- Given:
  - Set of facts: KB
  - Question: β
- Does KB *entail* β: if KB is true, must β be true?
  - KB $\models$ β

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- Example:

$$\{P, P \Rightarrow Q\} \models Q$$

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- Given:
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- Example:

$$\underbrace{\boxed{\{P, P \Rightarrow Q\}}}_{KB} \models \underbrace{\boxed{Q}}_{\beta}$$

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P	Q	$P \Rightarrow Q$
true	true	true
true	false	false
false	true	true
false	false	true

Propositional Logic: Semantics

- Given:
  - Set of facts: KB
  - Question: β
- Does KB *entail* β: if KB is true, must β be true?
  - $KB \vDash \beta$
- Example:

$\{P, P \Rightarrow Q\} \vDash Q$

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  - Question: β
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  - KB ⊨ β
- Example:

$\{P, P \Rightarrow Q\} \vDash Q$

P	Q	$P \Rightarrow Q$
true	true	true
true	false	false
false	true	true
false	false	true

Propositional Logic: Semantics

- Given:
  - Set of facts: KB
  - Question: β
- Does KB *entail* β: if KB is true, must β be true?
  - KB $\vDash$ β
- Example:

$\{P, P \Rightarrow Q\} \vDash Q$

P	Q	$P \Rightarrow Q$
true	true	true
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false	true	true
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Propositional Logic: Semantics

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P	Q	$P \Rightarrow Q$
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Propositional Logic: Semantics

- Given:
  - Set of facts: KB
  - Question: β
- Does KB *entail* β: if KB is true, must β be true?
  - KB $\models$ β
- Example:

$$\left\{\begin{array}{l} \text{mythical} \Rightarrow \text{immortal} \\ \neg\,\text{mythical} \Rightarrow (\text{mortal} \wedge \text{mammal}) \\ (\text{immortal} \vee \text{mammal}) \Rightarrow \text{horned} \\ \text{horned} \Rightarrow \text{magical} \end{array}\right\} \models \text{magical}$$

Propositional Logic: Semantics

- Given:
  - Set of facts: KB
  - Question: β
- Does KB *entail* β: if KB is true, must β be true?
  - KB $\models \beta$

KB

- Example: $\boxed{\begin{array}{l}\{\text{mythical} \Rightarrow \text{immortal} \\ \neg\, \text{mythical} \Rightarrow (\text{mortal} \wedge \text{mammal}) \\ (\text{immortal} \vee \text{mammal}) \Rightarrow \text{horned} \\ \text{horned} \Rightarrow \text{magical}\}\end{array}}$ $\models \boxed{\text{magical}}$

β

Propositional Logic: Semantics

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- Does KB *entail* β: if KB is true, must β be true?
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- Example: $\boxed{\begin{array}{l}\{\text{mythical} \Rightarrow \text{immortal} \\ \neg\, \text{mythical} \Rightarrow (\text{mortal} \wedge \text{mammal}) \\ (\text{immortal} \vee \text{mammal}) \Rightarrow \text{horned} \\ \text{horned} \Rightarrow \text{magical}\}\end{array}}$ $\models \boxed{\text{magical}}$

β

Truth Table would answer this – if every "line" where all of KB is true β is also true

Propositional Logic: Semantics

MOST COMMON CONFUSION

$\varphi \Rightarrow \psi$ is a sentence IN logic

$\varphi \vDash \psi$ is a statement ABOUT sentences in logic

Propositional Logic: Semantics

MOST COMMON CONFUSION

$\varphi \Rightarrow \psi$ is a sentence IN logic

$P \Rightarrow Q$

$\varphi \vDash \psi$ is a statement ABOUT sentences in logic

$\{P, (P \Rightarrow Q)\} \vDash Q$

Propositional Logic: Semantics

MOST COMMON CONFUSION

$\varphi \Rightarrow \psi$ is a sentence IN logic

$P \Rightarrow Q$

$\varphi \vDash \psi$ is a statement ABOUT sentences in logic

$\{P, (P \Rightarrow Q)\} \vDash Q$

Gets confusing, because $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ (a sentence in logic) is always true

Special Forms of Entailment

- Tautology, Valid: Is α true for all truth assignments

$$\models \alpha$$

Special Forms of Entailment

- Tautology, Valid: Is α true for all truth assignments

$\models \alpha$ Example: $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$

Special Forms of Entailment

- Tautology, Valid: Is α true for all truth assignments

$$\models \alpha$$

Example: $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$

- Unsatisfiable: Is α false for all truth assignments

$$\alpha \models \text{False}$$

$$\alpha \models ()$$

Special Forms of Entailment

- Tautology, Valid: Is α true for all truth assignments

$\vDash \alpha$ Example: $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$

- Unsatisfiable: Is α false for all truth assignments

$\alpha \vDash \text{False}$ Example: $P \wedge \neg P$

$\alpha \vDash ()$

Special Forms of Entailment

- Tautology, Valid: Is α true for all truth assignments

  $\models \alpha$ Example: $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$

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  $\alpha \models ()$

- Satisfiable: Does there exist a truth assignment for which α is true

  Equivalent to: $\neg\alpha$ is not valid

Special Forms of Entailment

- Tautology, Valid: Is α true for all truth assignments

  $\models \alpha$ Example: $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$

- Unsatisfiable: Is α false for all truth assignments

  $\alpha \models \text{False}$ Example: $P \wedge \neg P$

  $\alpha \models ()$

- Satisfiable: Does there exist a truth assignment for which α is true

  Equivalent to: $\neg\alpha$ is not valid

  Example: $P \wedge Q$

Propositional Logic: Semantics

KB $\models$? β

1. mythical $\Rightarrow$ immortal
2. $\neg$mythical $\Rightarrow$ (mortal $\wedge$ mammal)
3. (immortal $\vee$ mammal) $\Rightarrow$ horned
4. horned $\Rightarrow$ magical

magical

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- Syntax: What you can write down
- Semantics: The connection between what you write down and their meaning in the world being represented
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Propositional Logic: Inference

- Given:
  - Set of facts: KB
  - Question: β
- Can β be inferred from KB using a set of inference rules I:
  Can rules in I allow you to conclude β when starting from KB?
  Does KB *imply* β using I?
  - $KB \vdash_{\mathrm{I}} \beta$
  - I is usually clear from context, so usually just write $KB \vdash \beta$

Propositional Logic: Semantics

KB	$\vDash$?	β
1. mythical $\Rightarrow$ immortal 2. $\neg$mythical $\Rightarrow$ (mortal $\wedge$ mammal) 3. (immortal $\vee$ mammal) $\Rightarrow$ horned 4. horned $\Rightarrow$ magical		magical

Propositional Logic: Inference

KB $\vdash$? β

1. mythical $\Rightarrow$ immortal
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magical

Propositional Logic: Inference

- Could answer this using truth tables:
  - For every set of truth assignments for which KB is true is β also true?

- Example:

  Find all combinations of

  mythical $\in$ {true,false}
  immortal $\in$ {true,false}
  mortal $\in$ {true,false}
  mammal $\in$ {true,false}
  horned $\in$ {true,false}
  magical $\in$ {true,false}

  for which

  mythical $\Rightarrow$ immortal
  $\neg$ mythical $\Rightarrow$ (mortal $\wedge$ mammal)
  (immortal $\vee$ mammal) $\Rightarrow$ horned
  horned $\Rightarrow$ magical)

  are true, then see if magical is true in all of them

- Problem: Exponential in number of variables

Propositional Logic: Inference

- Solution: Use inference rules that let you conclude new sentences from existing sentences

- Examples: Modus Ponens, Conjunction Elimination

Propositional Logic: Syntax vs Semantics vs Inference

MOST COMMON CONFUSION

$\varphi \Rightarrow \psi$ is a statement in logic

$\varphi \vDash \psi$ is a statement about logic semantics

$\varphi \vdash \psi$ is a statement about specific logic inference rules

Propositional Logic: Syntax vs Semantics vs Inference

MOST COMMON CONFUSION

$\varphi \Rightarrow \psi$ is a statement in logic

$\varphi \vDash \psi$ is a statement about logic semantics

$\varphi \vdash \psi$ is a statement about specific logic inference rules

(CS motive: use syntactic rules mechanically to infer new entailed sentences)

Why Propositional Logic?

There exist inference rules I such that
if φ and ψ are in propositional logic:

$\varphi \vDash \psi$ if an only if $\varphi \vdash_{\mathrm{I}} \psi$

Why Propositional Logic?

There exist inference rules I such that
if φ and ψ are in propositional logic:

$\varphi \vDash \psi$ if an only if $\varphi \vdash_{\mathrm{I}} \psi$

One direction (if $\varphi \vdash_{\mathrm{I}} \psi$ then $\varphi \vDash \psi$) says:
anything you infer is true ("soundness")

The other direction(if $\varphi \vDash \psi$ then $\varphi \vdash_{\mathrm{I}} \psi$) says:
anything true can be inferred ("completeness")

Soundness and Completeness of Inference Rules

- Soundness: Anything that you conclude must be true

$$KB \vdash \beta \quad \text{implies} \quad KB \models \beta$$

- Completeness: If something is true it can be inferred by the rules

$$KB \models \beta \quad \text{implies} \quad KB \vdash \beta$$

Our Focus: Resolution Theorem Proving

Sound and complete for propositional logic*

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Sound and complete for propositional logic*

*requires more details

Resolution Theorem Proving

Detail 1:

$$KB \models \beta$$

if and only if

$$KB \wedge \neg (\beta) \text{ is unsatisfiable}$$

Resolution Theorem Proving

Detail 1:

$$KB \vDash \beta$$

if and only if

$KB \wedge \neg (\beta)$ is unsatisfiable

$$KB \wedge \neg (\beta) \vDash ()$$

Resolution Theorem Proving

Detail 1:

$$KB \models \beta$$

if and only if

$KB \wedge \neg (\beta)$ is unsatisfiable

$$KB \wedge \neg (\beta) \models ()$$

Resolution is complete only for unsatisfiability

Resolution Theorem Proving

Detail 1:

$$KB \vDash \beta$$

if and only if

$KB \wedge \neg (\beta)$ is unsatisfiable

$$KB \wedge \neg (\beta) \vDash ()$$

Resolution is complete only for unsatisfiability

(“Refutation complete”)

Resolution Theorem Proving

Detail 2:

Requires that all sentences be written in a constrained subset of propositional logic called

“Conjuctive (or Clausal) Normal Form”

(CNF)

Resolution Theorem Proving

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Requires that all sentences be written in a constrained subset of propositional logic called

“Conjuctive (or Clausal) Normal Form”

(CNF)

Every sentence in propositional logic has an equivalent sentence in CNF
(“equivalent” means truth table is identical)

Conjunctive Normal Form (CNF)

- CNF: Conjunction of disjunction of literals

<Literal> = <PropositionalSymbol> | $\neg$ <PropositionalSymbol>
<Clause> = (<Literal$_1$> $\vee$ … $\vee$ <Literal$_k$>) | ()
<Sentence> = (<Clause$_1$> $\wedge$ … <Clause$_n$>)

Conjunctive Normal Form (CNF)

- CNF: Conjunction of disjunction of literals

  <Literal> = <PropositionalSymbol> | $\neg$ <PropositionalSymbol>
  <Clause> = (<Literal$_1$> $\vee$ … $\vee$ <Literal$_k$>) | ()
  <Sentence> = (<Clause$_1$> $\wedge$ … <Clause$_n$>)

- Any sentence α in propositional logic has an equivalent sentence CNF(α) in CNF

$$\alpha \vDash \mathrm{CNF}(\alpha)$$

$$\mathrm{CNF}(\alpha) \vDash \alpha$$

Resolution Theorem Proving

To determine

$$\mathrm{KB} \models ?\ \beta$$

Do resolution theorem proving

$$\mathrm{CNF}(\mathrm{KB}) \wedge \mathrm{CNF}(\neg\beta) \vdash ?\ ()$$

Resolution Theorem Proving

Algorithm:

1. Convert KB and $\neg$ (β) to CNF, conjoin them to get KB’
2. Apply resolution to KB’ until you get an empty conjunction

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$$\overbrace{\boxed{\begin{array}{l}\{\text{mythical} \Rightarrow \text{immortal} \\ \neg\ \text{mythical} \Rightarrow (\text{mortal} \wedge \text{mammal}) \\ (\text{immortal} \vee \text{mammal}) \Rightarrow \text{horned} \\ \text{horned} \Rightarrow \text{magical}\}\end{array}}}^{\text{KB}} \vDash ? \overbrace{\boxed{\text{magical}}}^{\beta}$$

- Example: {mythical ⇒ immortal, ¬ mythical ⇒ (mortal ∧ mammal), (immortal ∨ mammal) ⇒ horned, horned ⇒ magical} ⊨? magical

KB

- Example: $[(\text{mythical} \Rightarrow \text{immortal}) \land$
  $(\neg\, \text{mythical} \Rightarrow (\text{mortal} \land \text{mammal})) \land$
  $((\text{immortal} \lor \text{mammal}) \Rightarrow \text{horned}) \land$
  $\text{horned} \Rightarrow \text{magical}]$ $\vDash ?$ magical

β

Example

$$(\text{mythical} \Rightarrow \text{immortal}) \wedge (\neg\text{mythical} \Rightarrow (\text{mortal} \wedge \text{mammal}))$$
$$\wedge ((\text{immortal} \vee \text{mammal}) \Rightarrow \text{horned}) \wedge (\text{horned} \Rightarrow \text{magical}) \vDash \text{magical} \ ?$$

Example

$(\text{mythical} \Rightarrow \text{immortal}) \wedge (\neg\text{mythical} \Rightarrow (\text{mortal} \wedge \text{mammal}))$
$\quad \wedge ((\text{immortal} \vee \text{mammal}) \Rightarrow \text{horned}) \wedge (\text{horned} \Rightarrow \text{magical}) \vDash \text{magical}$?

$(\text{my} \Rightarrow \text{i}) \wedge (\neg\text{my} \Rightarrow (\text{mo} \wedge \text{ml})) \wedge ((\text{i} \vee \text{ml}) \Rightarrow \text{h}) \wedge (\text{h} \Rightarrow \text{ma}) \vDash \text{ma}$?

Compute CNF($(\text{my} \Rightarrow \text{i}) \wedge (\neg\text{my} \Rightarrow (\text{mo} \wedge \text{ml})) \wedge ((\text{i} \vee \text{ml}) \Rightarrow \text{h}) \wedge (\text{h} \Rightarrow \text{ma})$)
Compute CNF($\neg\text{ma}$)

Conversion to CNF
Step 1: Remove $\Rightarrow$

$\alpha \Rightarrow \beta$ is the same as $\neg\alpha \vee \beta$

1. Replace some $\alpha \Rightarrow \beta$ with $\neg\, \alpha \vee \beta$

   Repeat until no $\Rightarrow$ is left

[no more "$\Rightarrow$"]

Example

$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge\; (h \Rightarrow ma)$

Example

$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge\; (h \Rightarrow ma)$

Example

$\underline{(my \Rightarrow i)} \wedge \underline{(\neg my \Rightarrow (mo \wedge ml))} \wedge \underline{((i \vee ml) \Rightarrow h)} \wedge \underline{(h \Rightarrow ma)}$

$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\neg(i \vee ml) \vee h) \wedge (\neg h \vee ma)$

Conversion to CNF
Step 2: Push in $\neg$

$\neg(\alpha \vee \beta)$ is the same as $\neg\alpha \wedge \neg\beta$

$\neg(\alpha \wedge \beta)$ is the same as $\neg\alpha \vee \neg\beta$

(DeMorgan's Laws)

2. Replace $\neg(\alpha \vee \beta)$ with $\neg\alpha \wedge \neg\beta$ or Replace $\neg(\alpha \wedge \beta)$ with $\neg\alpha \vee \neg\beta$

Repeat until no more can be done

[no more "$\Rightarrow$"]

["$\neg$" only before propositional symbols]

Example

$$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge \;(h \Rightarrow ma)$$

Step 1 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\neg(i \vee ml) \vee h) \wedge (\neg h \vee ma)$$

Example

$$(my \Rightarrow i) \land (\neg my \Rightarrow (mo \land ml)) \land ((i \lor ml) \Rightarrow h) \land (h \Rightarrow ma)$$

Step 1 gives

$$(\neg my \lor i) \land (\neg\neg my \lor (mo \land ml)) \land (\underline{\neg(i \lor ml)} \lor h) \land (\neg h \lor ma)$$

Example

$$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge \;(h \Rightarrow ma)$$

Step 1 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\underline{\neg(i \vee ml)} \vee h) \wedge (\neg h \vee ma)$$

Step 2 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$$

Example

$$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge\; (h \Rightarrow ma)$$

Step 1 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\neg(i \vee ml) \vee h) \wedge (\neg h \vee ma)$$

Step 2 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$$

Conversion to CNF
Step 3: Remove Double-Negation

$\neg\neg\alpha$ is the same as α

3. Replace $\neg\neg\alpha$ with α

Repeat until no more can be done

[no more "$\Rightarrow$"]

["$\neg$" only before propositional symbols]

["$\neg$" only appears as singleton]

Example

$$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge \; (h \Rightarrow ma)$$

Step 1 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\neg(i \vee ml) \vee h) \wedge (\neg h \vee ma)$$

Step 2 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$$

Example

$$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge \;(h \Rightarrow ma)$$

Step 1 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\neg(i \vee ml) \vee h) \wedge (\neg h \vee ma)$$

Step 2 gives

$$(\neg my \vee i) \wedge (\underline{\neg\neg my} \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$$

Example

$$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge \; (h \Rightarrow ma)$$

Step 1 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\neg(i \vee ml) \vee h) \wedge (\neg h \vee ma)$$

Step 2 gives

$$(\neg my \vee i) \wedge (\underline{\neg\neg my} \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$$

Step 3 gives

$$(\neg my \vee i) \wedge (my \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$$

Example

$(my \Rightarrow i) \wedge (\neg my \Rightarrow (mo \wedge ml)) \wedge ((i \vee ml) \Rightarrow h) \wedge (h \Rightarrow ma)$

Step 1 gives

$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\neg(i \vee ml) \vee h) \wedge (\neg h \vee ma)$

Step 2 gives

$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$

Step 3 gives

$(\neg my \vee i) \wedge (my \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$

Conversion to CNF
Step 4: Distribute $\vee$ over $\wedge$

$\alpha \vee (\beta \wedge \gamma)$ is the same as $(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

4. Replace $\alpha \vee (\beta \wedge \gamma)$ with $(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

Repeat until no more can be done

[no more "$\Rightarrow$"]

["$\neg$" only before propositional symbols]

["$\neg$" only appears as singleton]

[done]

Example

$$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge \; (h \Rightarrow ma)$$

Step 1 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\neg(i \vee ml) \vee h) \wedge (\neg h \vee ma)$$

Step 2 gives

$$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$$

Step 3 gives

$$(\neg my \vee i) \wedge (my \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$$

Example

$(my \Rightarrow i) \;\wedge\; (\neg my \Rightarrow (mo \wedge ml)) \;\wedge\; ((i \vee ml) \Rightarrow h) \wedge\; (h \Rightarrow ma)$

Step 1 gives

$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge (\neg(i \vee ml) \vee h) \wedge (\neg h \vee ma)$

Step 2 gives

$(\neg my \vee i) \wedge (\neg\neg my \vee (mo \wedge ml)) \wedge ((\neg i \wedge \neg ml) \vee h) \wedge (\neg h \vee ma)$

Step 3 gives

$(\neg my \vee i) \wedge (\underline{my \vee (mo \wedge ml)}) \wedge (\underline{(\neg i \wedge \neg ml) \vee h}) \wedge (\neg h \vee ma)$

Step 4 gives

$(\neg my \vee i) \wedge (my \vee mo) \wedge (my \vee ml) \wedge (\neg i \vee h) \wedge (\neg ml \vee h) \wedge (\neg h \vee ma)$

Example

$$\text{CNF}(\ (my \Rightarrow i)\ \wedge\ (\neg my \Rightarrow (mo \wedge ml))\ \wedge\ ((i \vee ml) \Rightarrow h) \wedge\ (h \Rightarrow ma)\ )$$

$$=$$

$$(\neg my \vee i) \wedge (my \vee mo) \wedge (my \vee ml) \wedge (\neg i \vee h) \wedge (\neg ml \vee h) \wedge (\neg h \vee ma)$$

Conversion to CNF Summary

1. Remove implications:
   Replace $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$ everywhere [no more "$\Rightarrow$"]
2. Push in $\neg$
   Replace $\neg(\alpha \vee \beta)$ with $\neg\alpha \wedge \neg\beta$
   Replace $\neg(\alpha \wedge \beta)$ with $\neg\alpha \vee \neg\beta$
   ["$\neg$" only before propositional symbols]
3. Remove double negations:
   Replace $\neg\neg\alpha$ with α ["$\neg$" only singletons]
4. Distribute $\vee$ over $\wedge$:
   Rewrite $\alpha \vee (\beta \wedge \gamma)$ as $(\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

Resolution Theorem Proving

To determine: KB $\vDash$? β

Algorithm:

1. Convert KB and $\neg$ β to CNF, conjoin them to get KB'
2. Apply resolution to KB' until you get an empty clause

Resolution Theorem Proving

To determine: KB $\vDash$? β

Algorithm:

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Resolution Inference Rule

$$\frac{(\boldsymbol{p} \vee \alpha_1 \vee \ldots \vee \alpha_a) \qquad (\neg \boldsymbol{p} \vee \beta_1 \vee \ldots \vee \beta_b)}{(\alpha_1 \vee \ldots \vee \alpha_a \vee \beta_1 \vee \ldots \vee \beta_b)}$$

where each α_i and β_j are literals (symbols or their negations) and $\boldsymbol{p}$ is a propositional symbol

Resolution Inference Rule

$$\frac{\begin{array}{c}(\boldsymbol{p} \vee \alpha_1 \vee \ldots \vee \alpha_a) \\ (\neg \boldsymbol{p} \vee \beta_1 \vee \ldots \vee \beta_b)\end{array}}{(\alpha_1 \vee \ldots \vee \alpha_a \vee \beta_1 \vee \ldots \vee \beta_b)}$$

where each α_i and β_j are literals (symbols or their negations) and $\boldsymbol{p}$ is a propositional symbol

This really means

$$\frac{\begin{array}{c}(\alpha_1 \vee \ldots \vee \alpha_{i-1} \vee \boldsymbol{p} \vee \alpha_{i+1} \vee \ldots \vee \alpha_a) \\ (\beta_1 \vee \ldots \vee \beta_{j-1} \vee \neg \boldsymbol{p} \vee \beta_{j+1} \vee \ldots \vee \beta_b)\end{array}}{(\alpha_1 \vee \ldots \vee \alpha_{i-1} \vee \alpha_{i+1} \vee \ldots \vee \alpha_a \vee \beta_1 \vee \ldots \vee \beta_{j-1} \vee \beta_{j+1} \vee \ldots \vee \beta_b)}$$

(but this becomes more awkward notationally)

Resolution Inference Rule Example

$(P \lor Q \lor \neg R)$

$\underline{(\neg S \lor T \lor \neg Q \lor V)}$

Resolution Inference Rule Example

$$(P \lor Q \lor \neg R)$$
$$\underline{(\neg S \lor T \lor \neg Q \lor V)}$$

Resolution Inference Rule Example

$$\frac{\begin{array}{c}(P \vee Q \vee \neg R) \\ (\neg S \vee T \vee \neg Q \vee V)\end{array}}{(P \vee \neg R \vee \neg S \vee T \vee V)}$$

Resolution Theorem Proving

To determine: KB $\vDash$? β

Algorithm:

1. Convert KB and $\neg\beta$ to CNF, conjoin them to get KB'
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Resolution Proof Example

$(my \Rightarrow i) \wedge (\neg my \Rightarrow (mo \wedge ml)) \wedge ((i \vee ml) \Rightarrow h) \wedge (h \Rightarrow ma) \vDash ma$?

Compute CNF($(my \Rightarrow i) \wedge (\neg my \Rightarrow (mo \wedge ml)) \wedge ((i \vee ml) \Rightarrow h) \wedge (h \Rightarrow ma)$)

Compute CNF($\neg ma$)

KB' = $(\neg my \vee i) \wedge (my \vee mo) \wedge (my \vee ml) \wedge (\neg i \vee h) \wedge (\neg ml \vee h) \wedge (\neg h \vee ma) \wedge \neg ma$

Resolution Proof Example

$(\neg\, my \lor i) \land (my \lor mo) \land (my \lor ml) \land (\neg i \lor h) \land (\neg ml \lor h) \land (\neg\, h \lor ma) \;\; \land \neg\, ma$

Resolution Proof Example

$(\neg\, my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $(\neg i \vee h)$ $(\neg ml \vee h)$ $(\neg\, h \vee ma)$ $(\neg\, ma)$

Resolution Proof Example

$(\neg my \lor i)$ $(my \lor mo)$ $(my \lor ml)$ $(\neg i \lor h)$ $(\neg ml \lor h)$ $(\neg h \lor ma)$ $(\neg ma)$

Resolution Proof Example

$(\neg\, my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $(\neg i \vee h)$ $(\neg ml \vee h)$ $(\neg\, h \vee ma)$ $(\neg\, ma)$

$(i \vee ml)$

Resolution Proof Example

$(\neg\ my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $(\neg i \vee h)$ $(\neg ml \vee h)$ $(\neg\ h \vee ma)$ $(\neg\ ma)$

$(i \vee ml)$

Resolution Proof Example

$(\neg\, my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $\underline{(\neg i \vee h)}$ $(\neg ml \vee h)$ $(\neg\, h \vee ma)$ $(\neg\, ma)$

$\underline{(i \vee ml)}$

Resolution Proof Example

$(\neg\, my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $\underline{(\neg i \vee h)}$ $(\neg ml \vee h)$ $(\neg\, h \vee ma)$ $(\neg\, ma)$

$\underline{(i \vee ml)}$

$(ml \vee h)$

Resolution Proof Example

$(\neg my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $(\neg i \vee h)$ $(\neg ml \vee h)$ $(\neg h \vee ma)$ $(\neg ma)$

$(i \vee ml)$

$(ml \vee h)$

Resolution Proof Example

$(\neg\, my \lor i)$ $(my \lor mo)$ $(my \lor ml)$ $(\neg i \lor h)$ $\underline{(\neg ml \lor h)}$ $(\neg\, h \lor ma)$ $(\neg\, ma)$

$(i \lor ml)$

$\underline{(ml \lor h)}$

Resolution Proof Example

$(\neg\, my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $(\neg i \vee h)$ $\underline{(\neg ml \vee h)}$ $(\neg\, h \vee ma)$ $(\neg\, ma)$

$(i \vee ml)$

$\underline{(ml \vee h)}$

(h)

Resolution Proof Example

$(\neg\, my \lor i)$ $(my \lor mo)$ $(my \lor ml)$ $(\neg i \lor h)$ $(\neg ml \lor h)$ $(\neg\, h \lor ma)$ $(\neg\, ma)$

$(i \lor ml)$

$(ml \lor h)$

(h)

Resolution Proof Example

$(\neg\, my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $(\neg i \vee h)$ $(\neg ml \vee h)$ $\underline{(\neg\, h \vee ma)}$ $(\neg\, ma)$

$(i \vee ml)$

$(ml \vee h)$

$(\underline{h)}$

Resolution Proof Example

$(\neg\, my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $(\neg i \vee h)$ $(\neg ml \vee h)$ $\underline{(\neg\, h \vee ma)}$ $(\neg\, ma)$

$(i \vee ml)$

$(ml \vee h)$

$\underline{(h)}$

(ma)

Resolution Proof Example

$(\neg\, my \lor i)$ $(my \lor mo)$ $(my \lor ml)$ $(\neg i \lor h)$ $(\neg ml \lor h)$ $(\neg\, h \lor ma)$ $(\neg\, ma)$

$(i \lor ml)$

$(ml \lor h)$

(h)

(ma)

Resolution Proof Example

$(\neg\, my \vee i)$ $(my \vee mo)$ $(my \vee ml)$ $(\neg i \vee h)$ $(\neg ml \vee h)$ $(\neg\, h \vee ma)$ $\underline{(\neg\, ma)}$

$(i \vee ml)$

$(ml \vee h)$

(h)

$\underline{(ma)}$

Resolution Proof Example

$(\neg\, my \lor i)$ $(my \lor mo)$ $(my \lor ml)$ $(\neg i \lor h)$ $(\neg ml \lor h)$ $(\neg\, h \lor ma)$ $\underline{(\neg\, ma)}$

$(i \lor ml)$

$(ml \lor h)$

(h)

$\underline{(ma)}$

$()$

Linearized Proof

1. $(\neg\, my \vee i)$
2. $(my \vee mo)$
3. $(my \vee ml)$
4. $(\neg i \vee h)$
5. $(\neg ml \vee h)$
6. $(\neg\, h \vee ma)$

Premises

7. $\neg\, ma$

Negated Goal

8.	$(i \vee ml)$	[1,3]
9.	$(ml \vee h)$	[8,4]
10.	h	[9,5]
11.	ma	[10,6]
12.	()	[11,7]