3. Propositional Encodings:

(a) Two different teams $h$ and $i$ cannot play in the same subslot $j$ in week $k$:

$$
\neg X_{hjk} \lor \neg X_{ijk}
$$

At least one team plays in subslot $j$ in week $k$:

$$
X_{1jk} \lor X_{2jk} \lor \cdots \lor X_{Njk}
$$

In summary we need the following clauses:

$$
\bigwedge_{1 \leq h, i, j \leq N} \, ^{1 \leq k < N}_{h \neq i} (\neg X_{hjk} \lor \neg X_{ijk}) \bigwedge_{1 \leq j \leq N} \, ^{1 \leq i < N}_{1 \leq k < N} (\lor X_{ijk})
$$

The number of clauses is $n(n - 1)\binom{n}{2} + n(n - 1) = O(n^4)$

(b)

$$
\bigwedge_{1 \leq i, j \leq N} \, ^{1 \leq k < N}_{j \neq l} (\neg X_{ijk} \lor \neg X_{iik})
$$

The number of clauses is $O(n^4)$.

Since each team plays at most once a week and exactly one team plays in each subslot, it can be easily seen that each team plays exactly once per week by the pigeon hole principle. If not, there must be some team(s) not playing some week(s). It turns out that in some subslots, there is no team playing in it, which is a contradiction with part a).

(c) We know that $X_{ihk} \land X_{j(h+1)k} \implies Y_{ijk}$ for different teams $i$ and $j$ in week $k$. (Adopt the convention that team $i$ plays in the odd numbered subslot and team $j$ plays in the even numbered subslot for variable $Y_{ijk}$.) Therefore, we have the following:

$$
\bigwedge_{1 \leq i, j \leq N} \, ^{1 \leq k, odd(h) < N}_{h \neq i} (\neg X_{ihk} \lor \neg X_{j(h+1)k} \lor Y_{ijk})
$$

However, we need another direction which states which $X$ variables could be true when variable $Y$ is true. We need the formulas:

$$
\left( Y_{ijk} \implies \bigvee_{1 \leq odd(h) < N} X_{ihk} \right) \land \left( \bigwedge_{1 \leq odd(h) < N} \left( (Y_{ijk} \land X_{ihk}) \implies X_{j(h+1)k} \right) \right)
$$

Note the first part says that if $Y_{ijk}$ is True then there has to be a team $i$ in one of the odd slots. The second part says: if $Y_{ijk}$ is True and team $i$ is in odd slot $h$, then team $j$ should be in even slot $h + 1$. 
Convert to CNF:

\[
\left( \neg Y_{ijk} \lor X_{ihk} \right) \land \left( \bigwedge_{1 \leq \text{odd}(h) < N} \left( \neg Y_{ijk} \lor \neg X_{ihk} \lor X_{j(h+1)k} \right) \right)
\]

Finally, AND all the clauses together:

\[
\bigwedge_{1 \leq i,j \leq N, \ 1 \leq k \leq N, \ j \neq i} \left( \left( \neg Y_{ijk} \lor X_{ihk} \right) \land \left( \bigwedge_{1 \leq \text{odd}(h) < N} \left( \neg Y_{ijk} \lor \neg X_{ihk} \lor X_{j(h+1)k} \right) \right) \right)
\]

** It is also possible to start with the following clause, but it is difficult to convert it to CNF:

\[
Y_{ijk} \implies \bigvee_{1 \leq \text{odd}(h) < N} \left( X_{ihk} \land X_{j(h+1)k} \right)
\]

(d) For this question, it is assumed that the previous sets of clauses exist, so you can assume that a team plays at most once a week.

\[
\bigwedge_{1 \leq i,j \leq N, \ 1 \leq k \leq N, \ k \neq i, i \neq j} \left( \neg Y_{ijk} \lor \neg Y_{ijl} \right) \land \left( \bigvee_{1 \leq k \leq N, \ 1 \leq i,j \leq N, \ i \neq j} Y_{ijk} \right)
\]

The first part says that teams \( i \) and \( j \) can’t play each other in more than one week. The second part says that \( i \) and \( j \) play each other in at least one of the weeks. Therefore, the AND of these two parts means that teams \( i \) and \( j \) play each other exactly once.

There are \( O(n^4) \) clauses.

(e) None of the above sets of clauses are redundant.

Without a), more than one team may play at the same subslot.

Without b), a team may play more than once a week.

Without c), variable \( Y \) is not well defined and d) has no meaning at all.

Without d), some pairs of teams may play each other more than once during the season.

(Certain clauses in the individual sets may be redundant because they can be inferred from other sets; however you cannot remove any set completely.)