Fourier Transforms
Fourier transform for 1D images

• A 1D image with N pixels is a vector of size N
• Every basis has N pixels
• There must be N basis elements
• n-th element of k-th basis in standard basis
  • \( E_k(n) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{otherwise} \end{cases} \)
• n-th element of k-th basis in Fourier basis
  • \( B_k(n) = e^{\frac{i2\pi kn}{N}} \)
A nuance

- $B_k(n) = e^{\frac{i2\pi kn}{N}}$

- We previously said as $k$ increases, frequency increases
  - i.e., more cycles within $N$

- $B_{N-k}(n) = e^{\frac{i2\pi (N-k)n}{N}} = e^{i2\pi n} - e^{\frac{i2\pi kn}{N}} = e^{-\frac{i2\pi kn}{N}} = \overline{B_k(n)}$ (complex conjugate)

- Frequency increases till $N/2$, subsequent basis elements are complex conjugates of previous elements
The full Fourier basis (N=10)

- Real
- Imaginary
- Basis number
- Highest frequency basis element
- Complex conjugates
The full Fourier basis (N=10)
A nuance

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• We previously said as \( k \) increases, frequency increases
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• \( B_{N-k}(n) = e^{\frac{i2\pi (N-k)n}{N}} = e^{i2\pi n - \frac{i2\pi kn}{N}} = e^{-\frac{i2\pi kn}{N}} = \overline{B_k(n)} = B_{-k}(n) \)

• Instead of considering \( B_0 \) to \( B_{N-1} \) as basis, often consider \( B_{\frac{N}{2}} \) to \( B_{\frac{N-1}{2}} \) as the basis
  • For odd \( N \), use \( B_{-(N-1)/2} \) to \( B_{(N-1)/2} \)
The full Fourier basis (N=10)
Fourier transform for 1D images

• Fourier transform converts from standard basis to Fourier basis
• If $x$ is image in standard basis, and $X$ is representation in Fourier basis, then:
  
  $X(k) = \sum_n x(n) e^{-\frac{i2\pi kn}{N}}$

• Or $X = B^* x$ where $B^*$ is a matrix with entries $B^*(k, n) = e^{-\frac{i2\pi kn}{N}}$
Inverse Fourier transform

• Given Fourier transform of image how should we get back image?
• In other words how should we change the basis back to the original coordinates?

\[ x = \sum_k X(k)B_k \]

\[ x(n) = \sum_k X(k)B_k(n) = \sum_k X(k)e^{\frac{i2\pi kn}{N}} \]
Real images in the Fourier basis

• Basis is complex but images are real
• Combine a pair of conjugate basis elements
• Coefficients for $B_k$ and $B_{-k}$ will be the same
  • $X(k) = X(-k)$
• So only need to know first $\frac{N}{2}$ Fourier coefficients!
The 0-th basis

• $B_k(n) = e^{\frac{i2\pi kn}{N}}$

• $B_0(n) = e^{\frac{i2\pi 0n}{N}} = 1$

• The zero-th basis element is a constant image

• Every other basis element varies between 1 and -1, so averages out to 0

• Thus, any image with a non-zero average intensity must have a high zero-th coefficient
Fourier transform for 2D images

• Images are 2D arrays
• Fourier basis for 1D array indexed by frequencies
• Fourier basis for 2D arrays are indexed by 2 spatial frequencies
• \((i,j)\)th Fourier basis for \(N \times N\) image
  • Has period \(N/i\) along \(x\)
  • Has period \(N/j\) along \(y\)

\[
B_{k,l}(x, y) = e^{\frac{2\pi ikx}{N} + \frac{2\pi ily}{N}}
= \cos \left(\frac{2\pi kx}{N} + \frac{2\pi lly}{N}\right) + i \sin \left(\frac{2\pi kx}{N} + \frac{2\pi lly}{N}\right)
\]
Visualizing the Fourier basis for images

$B_{1,1}$

$B_{3,20}$
Visualizing the Fourier transform

- Fourier coefficients are complex
- Instead of visualizing complex numbers we look at the squared absolute value $|X(k, l)|^2$
- This is called the *power spectrum*
- There are $N \times N$ Fourier coefficients, so we can show this as an $N \times N$ image.
- Because of complex conjugates only the first $\frac{N}{2} \times \frac{N}{2}$ coefficients are unique
- Because of very high values display using log
Visualizing the Fourier transform
Visualizing the Fourier transform
Visualizing the Fourier transform

• High peak for $B_{0,0}$. Why?
• High values on the X axis and low values on the Y axis. Why?
Why Fourier transforms?

• Think of image in terms of low and high frequency information
• Low frequency: large scale structure, no details
• High frequency: fine structure
Why Fourier transforms?
What if we zero out all high y-frequency components?
What if we zero out all high frequencies?

Removing high frequency components looks like Gaussian / mean filtering. Is there more to this relationship?
Dual domains

• Image: Spatial domain

• Fourier Transform: Frequency domain
  • Amplitudes are called spectrum

• For any transformations we do in spatial domain, there are corresponding transformations we can do in the frequency domain

• *And vice-versa*
Dual domains

• **Convolution** in spatial domain = *Point-wise multiplication* in frequency domain
  - Suppose $h = f \ast g$
  - Suppose $H$ is the Fourier transform of $h$, similarly $F$ and $G$
  - Then $H(k, l) = F(k, l)G(k, l)$

• **Convolution** in frequency domain = *Point-wise multiplication* in spatial domain

• To understand action of a filter, look at its *Fourier transform*
Filter

Fourier transform (Power spectrum)
Filter

Fourier transform (Power spectrum)
Gaussian filtering

• Fourier transform of a Gaussian is another Gaussian!

• So convolving with a Gaussian in spatial domain = multiplying with Gaussian in frequency domain
  • High frequencies get zeroed out

• Higher the standard deviation in spatial domain = lower the std in frequency domain
  • More frequencies get zeroed out.